STABILITY ANALYSIS OF CYLINDRICAL VLASOV EQUILIBRIA

BY

ROBERT W. SHORT

A thesis submitted in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

(Physics)

at the

UNIVERSITY OF WISCONSIN-MADISON

1979

ACKNOWLEDGEMENTS

I would like to thank my thesis advisor, Professor Keith R. Symon, for his interest, encouragement, and advice over the course of this research.

The staff of the National Magnetic Fusion Energy Computer Center, particularly Kirby Fong, has given valuable advice and assistance on the computational aspects of the project. Gregory Benford deserves thanks for suggesting one of the applications of the work.

I am grateful to the National Science Foundation, the Plasma Physics group, and the Physics Department for their support of my graduate study. Financial support for this research was provided by USDOE.

I thank Linda Dolan for her speed, excellence, and patience in the typing of this thesis.

This thesis is dedicated to my father, L. W. Short, for his unfailing love and encouragement.

TABLE OF CONTENTS

INTRODUCTION

Plasmas are inherently very complex systems, and any attempt to understand and predict their behavior must include simplifications and approximations. In stability analysis we commonly make two types of approximations: dynamic and geometric. We replace the individual particle equations of motion by an equation of motion for a continuous fluid either in configuration space (MHD and other "fluid" models), or phase space ("kinetic" models such as the Vlasov and Fokker-Planck equations). And we often simplify the geometry by ignoring boundaries and regarding the plasma as filling all space homogeneously or varying only in one linear direction (the "local approximation"[(23)]). There is generally a trade-off between these types of approximation in that the more sophisticated the dynamics, the cruder the geometry, and vice-versa, in order that the resulting equations be solvable. Thus stability analysis of a toroidal system can be carried out with a fair degree of rigor only in a fluid approximation, such as MHD, while velocity space instabilities are calculated by kinetic theory in greatly simplified geometries. Only recently, with the availability of large and fast computers, has it been feasible to treat problems which are both kinetic and non-local.

We describe here a general method of stability analysis which may be applied to a large class of such problems, namely those which are described dynamically by the Vlasov equation, and geometrically by cylindrical symmetry. In Chapter I, we present the method for the simple case of the Vlasov-Poisson (electrostatic) equations, and apply the results to a calculation of the lower-hybrid-drift instability in a plasma with a rigid rotor distribution function, a problem which has been treated by Davidson[7] using a somewhat different method. In Chapter II the method is extended to the full Vlasov-Maxwell (electromagnetic) equations, and in Chapter III we apply these results to a calculation of the instability of the extraordinary electromagnetic mode in a relativistic E-layer interacting with a background plasma. The results of these calculations are compared to those of Striffler and Kammash,[12] who have treated this problem in the local approximation.

One of the most important aspects of the present work is the method of carrying out the integration of the perturbed distribution function over phase space. The approach used here was inspired by the work of Lewis and Symon,[1] who expand the perturbed distribution function in eigenfunctions of the Liouville operator. In Appendix A we show that the two approaches are equivalent.

I. ELECTROSTATIC CASE AND APPLICATION TO LOWER HYBRID DRIFT INSTABILITY

In this chapter, to illustrate the general approach as simply as possible, we consider a cylindrical plasma in the electrostatic approximation. This approximation is often referred to as the "low-beta" limit. However, it has been shown(2,3,4) that a more appropriate characterization of a plasma for which the electrostatic approximation is valid is $\omega_{pe} \ll ck$, where ω_{pe} is the electron plasma frequency, k is a typical wavenumber, and c is the velocity of light.

The cylindrical coordinate system to be used is shown in Fig. (1.1). The plasma column is taken to be infinitely long in the z-direction and azimuthally symmetric. Its axis of symmetry is taken as the z-axis of coordinates, and it is surrounded by a coaxial conducting cylinder of radius R. The purpose of introducing this cylinder is simply to make the radial mode numbers discrete; if we are dealing with a problem in which a conducting boundary plays no significant role, we may recover the continuum modes by taking the limit $R \to \infty$. To make the modes discrete in the z-direction as well, we impose periodic boundary conditions in the z-direction with periodicity length L_z. Again, this restriction is simply for mathematical convenience, and we may remove it by allowing $L_z \to \infty$.

The only non-ignorable coordinate is r, and so the equilibrium scalar and vector potentials are $\phi^\circ(r)$ and $\underline{A}^\circ(r)$, where the 0 denotes the equilibrium value. The equilibrium fields are then:

$$\underline{E}^\circ = E^\circ(r)\hat{r} \; ,$$
$$\underline{B}^\circ = B^\circ_\theta(r) + B^\circ_z(r)\hat{z} \; . \tag{1.1}$$

The equilibrium distribution functions will be functions of the particle constants of the motion:

$$f_{oj}(\underline{r},\underline{v}) = f_{oj}(H,P_\theta,P_z) \; . \tag{1.2}$$

Here the subscript "j" denotes particle species, and

$$H = \frac{m_j}{2}\,(v_r^2 + v_\theta^2 + v_z^2) + e_j\phi^\circ(r) \; , \tag{1.3}$$

$$P_\theta = m_j r v_\theta + \frac{e_j}{c}\,A^\circ_z(r) \; , \tag{1.4}$$

$$P_z = m_j v_z + \frac{e_j}{c}\,A^\circ_z(r) \; . \tag{1.5}$$

In the collisionless electrostatic approximation the plasma is described by the Vlasov-Poisson equations:

$$\left[\frac{\partial}{\partial t} + \underline{v}\cdot\nabla + \frac{e_j}{m_j}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\frac{\partial}{\partial\underline{v}}\right] f(\underline{r},\underline{v},t) = 0 \; , \tag{1.6}$$

$$\nabla^2\phi(\underline{r},t) = -\sum_j 4\pi e_j \int d^3v f(\underline{r},\underline{v},t) \; . \tag{1.7}$$

For purposes of stability analysis we linearize the above equations, writing

$$f(\underline{r},\underline{v},t) = f_o(\underline{r},\underline{v}) + f_1(\underline{r},\underline{v},t) \ ,$$

$$\phi(\underline{r},t) = \phi_o(\underline{r}) + \phi_1(\underline{r},t) \ .$$

Here f_1 and ϕ_1 are small perturbations to be added to the quantities of equations (1.1) and (1.2). (Since in the electrostatic approximation $\nabla \times \underline{E} = 0$, the magnetic field and the vector potential $\underline{A}$ are not perturbed.) The linearized equations then read

$$\begin{aligned}(\frac{\partial}{\partial t} + L_o)\, f_1(\underline{r},\underline{v},t) &= - \frac{e_j}{m_j} \underline{E}_1(\underline{r},t) \cdot \frac{\partial}{\partial \underline{v}} f_{oj}(\underline{r},\underline{v}) \\ &= \frac{e_j}{m_m} [\nabla\phi_1(\underline{r},t)] \cdot \frac{\partial}{\partial \underline{v}} f_{oj}(\underline{r},\underline{v}) \ ,\end{aligned} \tag{1.8}$$

$$\nabla^2\phi_1(\underline{r},t) = - \sum_j 4\pi e_j \int d^3v f_{ij}(\underline{r},\underline{v},t) \ , \tag{1.9}$$

where we have defined the equilibrium Liouville operator

$$L_o = \underline{v} \cdot \nabla + \frac{e_j}{m_j} [\underline{E}_o + \frac{1}{c} (\underline{v}\times\underline{B}_o)] \cdot \frac{\partial}{\partial \underline{v}} \ .$$

From (1.3) - (1.5) we have the relations

$$\frac{\partial H}{\partial \underline{v}} = m_j\underline{v} \quad , \quad \frac{\partial P_\theta}{\partial \underline{v}} = m_j r\hat{\theta} \quad , \quad \frac{\partial P_z}{\partial \underline{v}} = m_j\hat{z} \ .$$

Consequently (1.8) may be rewritten:

$$\left(\frac{\partial}{\partial t} + L_o\right) f_{1j} = e_j \left[\frac{\partial f_{oj}}{\partial H} \underline{v} + \frac{\partial f_{oj}}{\partial P_\theta} r\hat{\theta} + \frac{\partial f_{oj}}{\partial P_z} \hat{z}\right] \cdot \nabla\phi_1$$

$$= e_j \left[\frac{\partial f_{oj}}{\partial H} \underline{v} \cdot \nabla + \frac{\partial f_{oj}}{\partial P_\theta} \frac{\partial}{\partial \theta} + \frac{\partial f_{oj}}{\partial P_z} \frac{\partial}{\partial z}\right] \phi_1 \ . \tag{1.10}$$

We are interested in unstable modes, so we take the time dependence of all perturbed quantities to be of the form $e^{-i\omega t}$, with $\mathrm{Im}(\omega) > 0$. Thus we write

$$\phi_1(\underline{r},t) = \phi_1(\underline{r})\, e^{-i\omega t} \quad , \quad f_{1j}(\underline{r},\underline{v},t) = f_{1j}(\underline{r},\underline{v})\, e^{-i\omega t} \ .$$

The operator $(\frac{\partial}{\partial t} + L_o)$ in (1.10), acting on a function of phase space variables and time, is equivalent to the total (or comoving) time derivative. It represents the time derivative of the function as seen by a particle moving along an unperturbed trajectory (i.e., the trajectory it would follow in the equilibrium fields $\underline{E}_o$, $\underline{B}_o$ which appear in L_o). Thus we may write

$$\left(\frac{\partial}{\partial t} + L_o\right)[f_{1j}(\underline{r},\underline{v})\, e^{-i\omega t}] = \frac{d}{dt} \{f_{1j}[\underline{r}(t),\underline{v}(t)]\, e^{-i\omega t}\} \ , \tag{1.11}$$

where $\underline{r}(t)$, $\underline{v}(t)$ represent the unperturbed particle orbits. We may solve (1.10) for the first order distribution function by integrating over time:

$$f_{1j}(\underline{r},\underline{v})\, e^{-i\omega t} \doteq \int_{-\infty}^{t} dt' \frac{d}{dt'} \{f_{1j}[\underline{r}'(t'),\underline{v}'(t')]\, e^{-i\omega t'}\}$$

$$= e_j \int_{-\infty}^{t} dt'\, e^{-i\omega t'} \left[\frac{\partial f_{oj}}{\partial H} \frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta} \frac{\partial}{\partial \theta'} + \frac{\partial f_{oj}}{\partial P_z} \frac{\partial}{\partial z'}\right] \phi_1[\underline{r}'(t')] \ , \tag{1.12}$$

where $\underline{r}'(t')$, $\underline{v}'(t')$ represent the unperturbed trajectory of a particle as a function of the dummy variable t' and

$$\underline{r}'(t) = \underline{r} \quad , \quad \underline{v}'(t) = \underline{v} \ . \tag{1.13}$$

In obtaining Eq. (1.12), we have used the fact that $\underline{v}' \cdot \nabla = \frac{d}{dt'}$ when acting on a function of $\underline{r}'$.

As we have assumed periodicity in the z-direction with periodicity length L_z, we may resolve $\phi_1(\underline{r})$ into its Fourier components in the ignorable coordinates θ and z:

$$\phi_1(\underline{r}) = \sum_{\ell,k} \phi_{\ell,k}(r)\ e^{i(\ell\theta+kz)} \ , \tag{1.14}$$

where $k = \frac{2\pi n}{L_z}$, n an integer.

Using (1.12) and (1.13), Poisson's equation (1.7) becomes

$$\begin{aligned}(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}) \sum_{\ell,k} \phi_{\ell,k}\ (r)\ e^{i(\ell\theta+kz)} \\ = - \sum_j 4\pi e_j^2 \int d^3v \int_{-\infty}^{t} dt'\ e^{-i\omega(t'-t)} \\ \cdot [\frac{\partial f_{oj}}{\partial H}\frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta}\frac{\partial}{\partial\theta'} + \frac{\partial f_{oj}}{\partial P_z}\frac{\partial}{\partial z'}] \\ \cdot \sum_{\ell' k'} \phi_{\ell' k'}\ (r')\ e^{i(\ell'\theta'+k'z')} \ .\end{aligned} \tag{1.15}$$

Note that though the right side of (1.15) contains t, it is actually independent of the value of t. In fact, we could remove t altogether by defining a new variable $\tau = t' - t$ and replacing $\int_{-\infty}^{t} dt'$ by $\int_{-\infty}^{0} d\tau$. We retain the formal t "dependence", however, as it will be useful below.

To isolate one Fourier component on the left side of (1.15), we multiply by $\frac{1}{2\pi} e^{-i(\ell\theta+kz)}$ and integrate over θ and z. From Eq. (1.13), we see that the quantities $\theta' - \theta$ and $z' - z$ are independent of θ and z, respectively, since for fixed t' and t changing θ changes θ' by the same amount, and similarly for z. Using the identities

$$k'z' - kz = k'(z'-z) + (k'-k)z \ ,$$

$$\ell'\theta' - \ell\theta = \ell'(\theta'-\theta) + (\ell'-\ell)\theta \ ,$$

we obtain

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\ell^2}{r^2} - k^2\right) \phi_{\ell,k}(r) =$$

$$- \sum_j 4\pi e_j^2 \int d^3v \int_{-\infty}^{t} dt' e^{-i\omega(t'-t)} \left[\frac{\partial f_{oj}}{\partial H} \frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta} \frac{\partial}{\partial \theta'} + \frac{\partial f_{oj}}{\partial P_z} \frac{\partial}{\partial z'}\right]$$

$$\cdot \ \phi_{\ell,k}(r') \, e^{i[\ell(\theta'-\theta)+k(z'-z)]} \ . \tag{1.16}$$

Note from the above argument that the right side of (1.16) does not depend on the ignorable coordinates, even though they appear there.

Next, we wish to expand the radial dependence of the perturbed potential in eigenfunctions of the Laplacian:

$$\phi_{\ell,k}(r) = \sum_n \alpha_n \phi_n(r) \ , \tag{1.17}$$

where $\phi_n(r)$ satisfies the eigenvalue equation

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} - \frac{\ell^2}{r^2}\right) \phi_n(r) = -\lambda_n^2 \phi_n(r) \; . \tag{1.18}$$

Here we have suppressed the subscripts ℓ and k on ϕ_n. Thus we have

$$\phi_n(r) = A_n\, J_\ell(\lambda_n r) \; ,$$

where J_l is the ℓ^{th} order Bessel function of the first kind,

$$A_n = \frac{\sqrt{2}}{RJ_{\ell+1}(\lambda_n R)}$$

is a normalization constant, and λ_n is the n^{th} root of the equation $J_\ell(\lambda_n R) = 0$. The functions $\phi_n(r)$ then satisfy the orthonormality relation

$$\int_o^R drr\phi_{n'}(r)\; \phi_n(r) = \delta_{nn'} \; . \tag{1.19}$$

Substituting (1.18) into (1.16), multiplying by $r\phi_{n'}(r)$, and integrating over r yields

$$\sum_n \alpha_n(-\lambda_n^2 - k^2)\; \delta_{nn'} = -\sum_j 4\pi e_j^2 \int_o^R drr\phi_{n'}(r) \int d^3v \int_{-\infty}^t dt'\; e^{-i\omega(t'-t)}$$

$$\left[\frac{\partial f_{oj}}{\partial H}\frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta}\frac{\partial}{\partial \theta'} + \frac{\partial f_{oj}}{\partial P_z}\frac{\partial}{\partial z'}\right]$$

$$\cdot\; e^{i[\ell(\theta'-\theta)+k(z'-z)]} \sum_n \alpha_n \phi_n(r') \; . \tag{1.20}$$

This equation is a linear relation in the expansion coefficients

α_n, and we can write it as

$$\sum_n D_{nn'}(\omega)(\lambda_n^2+k^2)\ \alpha_{n'} = 0 ,$$

where

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int_0^R drr \int d^3v\phi_n(r)\ e^{-i(\ell\theta+kz-\omega t)}$$
$$\cdot \int_{-\infty}^{t} dt'\ e^{-i\omega t'}\ [\frac{\partial f_{oj}}{\partial H}\frac{d}{dt'} + i\ell\frac{\partial f_{oj}}{\partial P_\theta} + ik_z\frac{\partial f_{oj}}{\partial P_z}]$$
$$\cdot\ \phi_{n'}(r')\ e^{i(\ell\theta'+kz')} \tag{1.21}$$

and

$$k_n^2 \equiv \lambda_n^2 + k^2 .$$

To carry out the time integral in (1.21), we must determine the unperturbed orbits $\underline{r}'(t')$. The particle Hamiltonian does not depend on θ, z, or t, so H is a constant equal to the particle energy and we have

$$H(r,P_r,P_\theta,P_z) = E .$$

Since P_θ, P_z, and E are constants of the motion, we can solve for P_r as a function of r:

$$P_r(r) = P_r(r,E,P_\theta,P_z) .$$

This gives $\dot{r}$ as a function of r:

$$\dot{r} = \frac{\partial H}{\partial P_r} = \dot{r}(r) \ .$$

If the particle motion in r is bounded and non-asymptotic, then a particle which is at r_o at some time t must return to r_o at some latter time t + T. Since the Hamiltonian depends only on r, when the particle returns to r_o, it must have the same radial velocity $\dot{r}(r_o)$. Thus the motion in r is periodic with period $T(P_\theta, P_z, E)$. Now $\dot{\theta} = \frac{\partial H}{\partial P_\theta}$ depends on t only through r so it too must be periodic. Thus we can write

$$\theta(t) = \eta t + \tilde{\theta}(t) + \theta_o \ ,$$

where η is a constant, $\tilde{\theta}(r)$ is periodic with period T, and θ_o is an initial value chosen so $\tilde{\theta}(0) = 0$. Similarly, we have $z(t) = \sigma t + \tilde{z}(t) + z_o$, where σ is a constant, $\tilde{z}(t)$ is periodic with period T, and z_o is chosen so $\tilde{z}(t = 0) = 0$. Consequently we may write

$$\phi_n(r')e^{i(\ell\theta' + kz')} = A_n \ e^{i(\ell\eta + k\sigma)} + J_\ell(\lambda_n r')e^{i\ell\tilde{\theta}'} \ e^{ik\tilde{z}'} \ e^{i(\ell\theta_o + kz_o)} \ . \tag{1.22}$$

In order to perform the time integration in (1.21), we wish to express the function in (1.22) as a Fourier series in time. First consider the z term $e^{ik\tilde{z}'(t')}$. Since $\tilde{z}$ is periodic in time, we may write

$$\tilde{z}(t) = \sum_n b_n \sin n\Omega t \quad ; \quad \Omega = \frac{2\pi}{T(E,P_\theta,P_z)} \quad . \tag{1.23}$$

Using the identity

$$e^{ip\sin q} = \sum_{n=-\infty}^{\infty} J_n(p)\, e^{inq}$$

we have

$$e^{ik\tilde{z}(t)} = \prod_n e^{ikb_n \sin n\Omega t} = \prod_n \left(\sum_m J_m(kb_n)\, e^{imn\Omega t}\right) . \tag{1.24}$$

Thus we have expanded the function $e^{ik\tilde{z}(t)}$ in a Fourier series in time. Next we wish to represent $J_\ell(\lambda_n r)\, e^{i\ell\tilde{\theta}}$ in this form as well. Consider first the function $re^{i\tilde{\theta}}$; it is periodic, and so we can write it as a Fourier series:

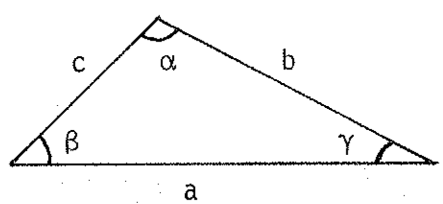
$$r(t)\, e^{i\tilde{\theta}(t)} = \sum_n a_n\, e^{i(\Omega_n t+\gamma_n)} , \tag{1.25}$$

where the a_n are chosen real, γ_n is a phase factor, and $\Omega_n = m_n\Omega$, m_n an integer and Ω given in Eq. (1.23). Here m_n is taken to be the frequency of the n^{th} largest term, so that the largest terms in the series occur first. In other words, instead of ordering the terms in the Fourier series by their frequency as is normally done, we order them by the magnitude of their coefficients. This is done solely for heuristic reasons, to simplify the representation of the particle orbits given below. To illustrate the approach, suppose we wish to take four terms of (1.24) as an adequate approximation to the particle motion. Then

$$re^{i\theta} = a_1\, e^{i(\Omega_1 t+\gamma_1)} + a_2\, e^{i(\Omega_2 t+\gamma_2)}$$
$$+ a_3\, e^{i(\Omega_3 t+\gamma_3)} + a_4\, e^{i(\Omega_4 t+\gamma_4)} \quad .$$

This function can be represented in the complex plane as in Fig. (1.2). Note that it corresponds to the particle motion in the x-y plane with the constant precession $e^{i\eta t}$ factored out.

Next we make use of Graf's theorem,[5] an addition theorem for Bessel functions, which states that for any triangle



we have

$$J_p(c)\; e^{ip\beta} = \sum_{m=-\infty}^{\infty} J_{p+m}(a)\; J_m(b)\; e^{im\gamma} \quad . \tag{1.26}$$

We apply this theorem repeatedly to the triangles in Fig. (1.2), obtaining

$$J_\ell(\lambda_r)\; e^{i\ell\tilde{\theta}} =$$

$$e^{i\ell(\Omega_1 t+\gamma_1)} \sum_{m_1} J_{\ell+m_1}(\lambda a_1)\; J_{m_1}(\lambda e) e^{-im_1[\alpha_1+(\Omega_2-\Omega_1)t+\gamma_2-\gamma_1]} (-1)^{m_1} \quad ,$$

$$J_{m_1}(\lambda e)\, e^{im_1\alpha_1} =$$

$$\sum_{m_2} J_{m_1+m_2}(\lambda a_2)\, J_{m_2}(\lambda f)\, e^{im_2[\alpha_2+(\Omega_3-\Omega_2)t+\gamma_3-\gamma_2]}\, (-1)^{m_2} ,$$

and finally

$$J_{m_2}(\lambda f)\, e^{im_2\alpha_2} =$$

$$\sum_{m_3} J_{m_2+m_3}(\lambda a_3)\, J_{m_3}(\lambda a_4)\, e^{-im_3[(\Omega_4-\Omega_3)t+\gamma_4-\gamma_3]}\, (-1)^{m_3} .$$

Combining these expressions we have

$$J_\ell(\lambda r)\, e^{i\ell\tilde{\theta}} =$$

$$\sum_{m_1,m_2,m_3} J_{\ell+m_1}(\lambda a_1)\, J_{m_1+m_2}(\lambda a_2)\, J_{m_2+m_3}(\lambda a_3)\, J_{m_3}(\lambda a_4)(-1)^{m_1+m_2+m_3}$$

$$e^{i[(\ell+m_1)(\Omega_1 t+\gamma_1)-(m_1+m_2)(\Omega_2 t+\gamma_2)+(m_2+m_3)(\Omega_3 t+\gamma_3)-m_3(\Omega_4 t+\gamma_4)]} . \tag{1.27}$$

Using (1.24) and (1.27), we can now write

$$\phi_n(r)\, e^{i(\ell\theta+kz)} = \sum_m G_m(E,P_\theta,P_z)\, e^{i(\ell\eta+k\sigma+m\Omega)t} , \tag{1.28}$$

where the coefficients G_m depend on the constants of the motion H, P_θ, and P_z through the coefficients a_n of (1.27) and b_n of (1.24). The best way to determine these coefficients, and the number needed for a good approximation to the particle orbit, will depend on the problem at hand. For the simple field configuration treated below, they may be found analytically; for a more complex equilibrium, it may be easiest to use a Galerkin method, substituting a certain number of terms of the series (1.24) and (1.27) into the equations of motion and solving the resulting algebraic equations for the a_n and b_n. In most cases of practical interest we would expect to need only a few epicycles, as in Fig. (1.2), to represent the particle motion well enough for a linearized stability analysis.

Using the expression (1.28) for the cylindrical harmonics we may rewrite (1.21) as

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int_0^R drr \int d^3v\phi_n(r)\, e^{-i(\ell\theta+kz-\omega t)} \int_{-\infty}^{t} dt'\, e^{-i\omega t'}$$

$$\cdot\, i\left[(\ell\eta+k\sigma+m\Omega)\frac{\partial f_{oj}}{\partial H} + \ell\,\frac{\partial f_{oj}}{\partial P_\theta} + k\,\frac{\partial f_{oj}}{\partial P_z}\right] \sum_m G_m\, e^{i(\ell\eta+k\sigma+m\Omega)t'} \;.$$

Note that θ_o and z_o cancel out of the exponentials due to the fact that the dispersion matrix is diagonal in ℓ and k. Remembering that $\mathrm{Im}(\omega) > 0$, we can perform the t' integration:

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int_0^R drr \int d^3v\phi_n(r)\, e^{-i(\ell\theta+kz)}$$

$$\cdot\, i\left[(\ell\eta+k_z\sigma + m\Omega)\,\frac{\partial f_{oj}}{\partial H} + \ell\,\frac{\partial f_{oj}}{\partial P_\theta} + k\,\frac{\partial f_{oj}}{\partial P_z}\right]$$

$$\cdot \sum_m G_m\, e^{i(\ell\eta+k\sigma+m\Omega)t} / i[\ell\eta+k\sigma+m\Omega-\omega] \;. \qquad (1.29)$$

Next we make use of the t "dependence" of the integrand in (1.29).

Writing the integrand as $I(r,\underline{v})$ we have

$$I(r,\underline{v}) = I(r,\underline{v},t) = I(r(t), \underline{v}(t))$$

where in the last equality $r(t)$ and $\underline{v}(t)$ represent the trajectory of a particle as a function of time. Varying t thus generates a curve in phase space along which I is invariant. We now use this fact to carry out part of the integration of I over the phase space variables. First we transform the variables of integration from the velocities to the canonical momenta:

$$\left|\frac{\partial(P_r,P_\theta,P_z)}{\partial(v_r,v_\theta,v_z)}\right| = m_j^3 r \rightarrow r dr d^3 v = \frac{1}{m_j^3} dr dP_r dP_\theta dP_z \ . \tag{1.30}$$

Next we wish to change the integration variables r, P_r to t, H.

$$dH = -\dot{P}_r dr + \dot{r} dP_r \quad , \quad dt = \frac{1}{\dot{r}} dt \ ,$$

so

$$\left|\frac{\partial(t,H)}{\partial(r,P_r)}\right| = 1 \quad , \quad dr dP_r = dt dH \ . \tag{1.31}$$

From (1.30) and (1.31) we have

$$r dr d^3 v = m_j^{-3} \, dt dH dP_\theta dP_z \ ,$$

so using (1.28), we can write (1.29) as

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int dH dP_\theta dP_z \int_0^T dt \sum_{m'} G^*_{m'}\, e^{-i(\ell\eta+k\sigma+m'\Omega)t}$$

$$\cdot \left[(\ell\eta+k\sigma+m\Omega)\frac{\partial f_{oj}}{\partial H} + \ell\frac{\partial f_{oj}}{\partial P_\theta} + k\frac{\partial f_{oj}}{\partial P_z}\right] \sum_m \frac{G_m e^{i(\ell\eta+k\sigma+m\Omega)t}}{\ell\eta+k\sigma+m\Omega-\omega}$$

$$= \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int dH dP_\theta dP_z T\,(H,P_\theta,P_z)$$

$$\cdot \sum_m \left[(\ell\eta+k\sigma+m\Omega)\frac{\partial f_{oj}}{\partial H} + \ell\frac{\partial f_{oj}}{\partial P_\theta} + k\frac{\partial f_{oj}}{\partial P_z}\right] \frac{|G_m(H,P_\theta,P_z)|^2}{\ell\eta+k\sigma+m\Omega-\omega} \quad . \tag{1.32}$$

Now we are left with integrals over the constants of the motion H, P_θ, P_z to perform in order to evaluate the elements of the dispersion matrix D. These must usually be carried out numerically. The normal modes of the system are found by solving

$$\det[\bar{D}(\omega)] = 0$$

for ω, where $\bar{D}$ represents a suitable truncation of D (i.e., one that includes enough of the expansion functions ϕ_n to represent the normal modes to the desired accuracy). If ω has a positive imaginary part, the corresponding mode is unstable, and the imaginary part of ω gives its growth rate.

To illustrate this procedure further, we next consider a simple class of problems for which the series (1.24) and (1.27) have only one term. We take the equilibrium magnetic field to be uniform and directed in the z-direction $\underline{B}_0 = B_0\hat{z}$, and the electric field to be radial and proportional to r: $\underline{E}_0 = E_0 r\hat{r}$, where B_0 and E_0 are constants. These fields give the equilibrium bulk motion of the plasma a "rigid-rotor" character, as explained below. The equation of motion for a particle in these fields is

$$m_m \frac{d\underline{v}}{dt} = e_j E_0 r\hat{r} + \frac{e_j}{c} \underline{v} \times B_0\hat{z} ,$$

or, in Cartesian coordinates

$$m_m \ddot{x} = e_m E_0 x + \frac{e_j B_0}{c} \dot{y} \quad , \quad m_j \ddot{y} = e_j E_0 y - \frac{e_j B_0}{c} \dot{x} \quad , \quad \ddot{z} = 0 .$$

If we let $\xi = x + iy$, we can write these equations as[6]

$$\ddot{\xi} + i\omega_{cj}\dot{\xi} - \frac{e_j}{m_j} E_0 \xi = 0 , \tag{1.33}$$

where $\omega_{cj} = \frac{e_j B_0}{m_j c}$, the cyclotron frequency for species j. This is a second order linear equation, and the solution may be written

$$\xi = ae^{i\omega_a t} + be^{i\omega_b t} , \tag{1.34}$$

where we may take a and b to be real and non-negative, so that

$$r(t=0) = a + b \quad , \quad y(t=0) = 0 .$$

Substituting (1.34) into (1.33), we determine ω_a and ω_b from the resulting algebraic equation:

$$\omega_{\substack{a\\b}} = -\frac{\omega_{cj}}{2} \pm \frac{1}{2}\sqrt{\omega_{cj}^2 - \frac{4e_j E_o}{m_j}} \quad . \tag{1.35}$$

For the particle motion to be bounded, we must impose the condition

$$\omega_{cj}^2 \geq \frac{4e_j E_o}{m_j} \quad .$$

Note that $\omega_a \geq \omega_b$.

The coefficients a and b in (1.33) determine the particle's radial position and its velocity, so they may be written in terms of H and P_θ. In practice, as shown below, it is more convenient to express H and P_θ in terms of a and b. In this equilibrium $\dot{z}$ is a constant of the motion, so $\tilde{z} = 0$ in (1.22) and $\sigma = \dot{z}$.

The particle motion in the x-y plane can be represented as in Fig. (1.3). Applying Graf's theorem (1.26) to the triangle (r, a, b), we have

$$\begin{aligned} \phi_n(r)\, e^{i\ell\theta} &= A_n\, e^{i\ell\omega_b t}\, J_\ell(\lambda_n r)\, e^{i\ell\tilde{\theta}} \\ &= A_n\, e^{i\ell\omega_b t} \sum_{m=-\infty}^{\infty} (-1)^m\, J_{\ell+m}(\lambda_n b)\, J_m(\lambda_n a)\, e^{-im(\omega_a-\omega_b)t} \quad . \end{aligned} \tag{1.36}$$

Substituting this expression into (1.21) and writing

$$D_{nn'}(\omega) = \delta_{nn'} + \sum_j \chi^j_{nn'}(\omega)$$

we have

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_0^R drr \int d^3v \sum_{m'=-\infty}^{\infty} (-1)^{m'} J_{\ell+m'}(\lambda_n b)\, J_m(\lambda_n a)$$

$$\cdot\, e^{-i[\ell\omega_b + kv_z - m'(\omega_a-\omega_b)-\omega]t} \int_{-\infty}^{t} dt' \sum_{m=-\infty}^{\infty}$$

$$\cdot \left\{ m_j \frac{\partial f_{oj}}{\partial H_\perp} [i\ell\omega_b - im(\omega_a-\omega_b)] + i\ell m_j \frac{\partial f_{oj}}{\partial P_\theta} + ik \frac{\partial f_{oj}}{\partial v_z} \right\}$$

$$(-1)^m J_{\ell+m}(\lambda_{n'} b)\, J_m(\lambda_{n'} a)\, e^{i[\ell\omega_b + kv_z - m(\omega_a-\omega_b)-\omega]t'} ,$$

where we have defined

$$H_\perp = \frac{m_j}{2}(v_r^2+v_\theta^2) + e_j\phi_o(r)$$

and replaced P_z by v_z using $P_z = m_j v_z$. We will refer to the χ's as "susceptibilities", since they contain the response of the plasma to the field. Since z is an ignorable coordinate, $H_\perp$ and v_z are also constants of the motion.

Doing the integral over t' we obtain

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_0^R drr \int d^3v \sum_{m'=-\infty}^{\infty} (-1)^{m'} J_{\ell+m'}(\lambda_n b)\, J_m(\lambda_n a)$$

$$\cdot\, e^{-i[\ell\omega_b + kv_z - m'(\omega_a-\omega_b)-\omega]t} \sum_{m=-\infty}^{\infty} \left\{ \frac{\partial f_{oj}}{\partial H_\perp} m_j[\ell\omega_b - m(\omega_a-\omega_b)]t + \ell m_j\right.$$

$$\left. \cdot\, \frac{\partial f_{oj}}{\partial P_\theta} + k\frac{\partial f_{oj}}{\partial v_z} \right\} (-1)^m J_{\ell+m}(\lambda_n b)\, J_m(\lambda_n a)\, \frac{e^{i[\ell\omega_b + kv_z - m(\omega_z-\omega_b)-\omega]t}}{\ell\omega_b + kv_z - m(\omega_s-\omega_b)-\omega} . \tag{1.37}$$

Using (1.30) and (1.31) we change the variables of integration from r, $\underline{v}$ to t, $H_\perp$, P_θ, and v_z and do the resulting integral over t to get

$$\chi^j_{nn'}(\omega) = \frac{4\pi e_j^2}{m_j^3 k_n^2} A_n A_{n'} \left[\frac{2\pi}{\omega_a - \omega_b}\right] \sum_m \int dH_\perp dP_\theta dv_z$$

$$\cdot \left\{ m_j \frac{\partial f_{oj}}{\partial H_\perp} [\ell\omega_b - m(\omega_a - \omega_b)] + \ell m_j \frac{\partial f_{oj}}{\partial v_z} + k \frac{\partial f_{oj}}{\partial v_z} \right\}$$

$$\cdot \frac{J_{\ell+m}(\lambda_n b) J_m(\lambda_n a) J_{\ell+m}(\lambda_{n'} b) J_m(\lambda_{n'} a)}{\omega + m(\omega_a - \omega_b) - kv_z - \ell\omega_b} , \quad (1.38)$$

where $\frac{2\pi}{\omega_a - \omega_b}$ appears as the period T of the motion in r.

For this equilibrium we have

$$H_\perp = \frac{m_j}{2} (v_r^2 + v_\theta^2) - \frac{1}{2} e_j E_o r^2 \quad , \quad P_\theta = m_j r v_\theta + \frac{1}{2} m_j \omega_{cj} r^2 \ .$$

When $r = a + b$, the particle is at its maximum in r, so that $v_r = 0$, $\theta = 0$, and $v = \omega_a a + \omega_b b$. Thus

$$H_\perp = \frac{1}{2} m_j (\omega_a a + \omega_b b)^2 - \frac{1}{2} e_j E_o (a+b)^2 \quad (1.39)$$

$$P_\theta = m_m (a+b)(\omega_a a + \omega_b b) + \frac{1}{2} m_j \omega_{cj} (a+b)^2 \ . \quad (1.40)$$

Equations (1.36) and (1.37) could be solved for a and b in terms of $H_\perp$ and P_θ and the results substituted into (1.35). However, the expressions for a and b in terms of $H_\perp$ and P_θ are considerably more complicated than (1.36) and (1.37), so it is more convenient to use

(1.36) and (1.37) to convert the $H_\perp$ and P_θ integrals in (1.35) to integrals over a and b. We have

$$\left|\frac{\partial(H_\perp,P_\theta)}{\partial(a,b)}\right| = m_j^2(\omega_a-\omega_b)\left|(\omega_a a+\omega_b b)^2 + (\omega_a a+\omega_b b)\omega_{cj}(a+b)+ \frac{e_j\varepsilon_o}{m_j}(a+b)^2\right| . \tag{1.41}$$

We shall now demonstrate that the expression inside the absolute value signs on the right of (1.41) is always negative, so that we may replace the absolute value signs with a factor of -1. Since $\omega_a - \omega_b > 0$, the expression can be positive only if

$$z^2 + \omega_{cj}z + \frac{e_jE_o}{m_j} \geq 0 , \tag{1.42}$$

where

$$z = \frac{\omega_a a+\omega_b b}{a+b} .$$

If we take the equality in (1.42), we have the same algebraic equation for z as that from which ω_a and ω_b were determined in Eqs. (1.33) - (1.35). Thus

$$z^2 + \omega_{cj}z + \frac{e_jE_o}{m_j} = 0 \quad \text{iff} \quad z = \omega_a \quad \text{or} \quad z = \omega_b ,$$

and the expression on the right of (1.42) can change sign only if z passes through one of the values ω_a or ω_b. But since a + b > 0, $\omega_a - \omega_b \geq 0$, we have

$$\omega_b \leq \frac{\omega_a a+\omega_b b}{a+b} \leq \omega_a \quad , \quad \omega_b \leq z \leq \omega_a \ . \tag{1.43}$$

Since the left side of (1.42) is clearly positive for $z \to \pm\infty$, and must change sign at $z = \omega_a$ or $z = \omega_b$, we see that for a in the range of (1.43) the left side of (1.42) must be non-positive. Therefore

$$\left|\frac{\partial(H_\perp,P_\theta)}{\partial(a,b)}\right| = -m_j^2(\omega_a-\omega_b)[(\omega_a a+\omega_b b)^2 + (\omega_a a+\omega_b b)\omega_{cj}(a+b)+ \frac{e_j E_o}{m_j}(a+b)^2]. \tag{1.44}$$

Using (1.44), (1.38) now becomes

$$\chi^j_{nn'}(\omega) = -\frac{8\pi^2 e_j^2}{m_j k_n^2} A_n A_{n'} \sum_{m=-\infty}^{\infty} \int_0^R da \int_0^{R-a} \int_{-\infty}^{\infty} dv_z$$

$$\cdot \left\{m_j \frac{\partial fo_j}{\partial H_\perp}[\ell\omega_b - m(\omega_a-\omega_b)] + \ell m_j \frac{\partial f_{oj}}{\partial P_\theta} + k_z \frac{\partial f_{oj}}{\partial v_z}\right\}$$

$$\cdot [(\omega_a a+\omega_b b)^2 + (\omega_a a+\omega_b b)\, \omega_{cj}(a+b) + \frac{e_j E_o}{m_j}(a+b)^2]$$

$$\cdot \frac{J_{\ell+m}(\lambda_n b)J_m(\lambda_n a)J_{\ell+m}(\lambda_{n'} b)J_m(\lambda_{n'} a)}{\omega+m(\omega_a-\omega_b)-k_z v_z-\ell\omega_b} \ . \tag{1.45}$$

The integrals over a and b in (1.45) have been taken over the range $a + b < R$ so that all orbits considered lie entirely within the cylinder of radius R. Strictly speaking, we can only allow equilibrium distribution functions f_{oj} which vanish identically outside the cylinder. This is because orbits extending outside R will not be treated correctly by the integration over unperturbed orbits, since the expansion of the perturbed potential is not valid there. In many cases, however, it is most convenient to choose a simple analytic function for f_{oj} and allow the limits of the phase space integration to exclude the unwanted orbits. This is more easily accomplished when the matrix elements are written in the form (1.45) than in the form (1.38), since the region in $H_\perp$, P_θ space which corresponds to orbits lying entirely inside the cylinder is much more complicated in structure than the simple triangle in a, b space which is integrated over in (1.45).

Having calculated the susceptibilities $\chi^j_{nn'}(\omega)$ in (1.45), the stability analysis is completed by solving the equation

$$\det[\delta_{nn'} + \sum_j \chi^j_{nn'}(\omega)] = 0$$

for ω using a suitable truncation of the dispersion matrix. How to determine a suitable truncation will be discussed further in Chapter III. The roots ω of this equation with positive imaginary part γ correspond to the unstable modes of the plasma, with γ being the growth rate.

R. C. Davidson[7] has presented a similar method of stability analysis for the special case when f_{oj} is a rigid rotor equilibrium. A rigid rotor equilibrium is characterized by a distribution function of the form

$$f_{oj}(\underline{r}\ \underline{v}) = f_{oj}(H_\perp - \omega_j P_\theta, v_z) \ ,$$

i.e., f_{oj} depends on $H_\perp$ and P_θ only through the linear combination $H_\perp - \omega_j P_\theta$, where ω_j is a constant for each species. It can be shown that in a reference frame rotating about the z-axis with frequency ω_j the rigid rotor distribution is isotropic in velocity space. Thus in the multi-fluid limit each species appears to rotate in bulk with angular frequency ω_j, and this gives rise to the name "rigid rotor."

For a rigid rotor distribution function we have

$$\frac{\partial f_{oj}}{\partial P_\theta} = -\omega_j \frac{\partial f_{oj}}{\partial H_\perp} \quad ,$$

and (1.38) becomes

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j^3 k_n^2} A_n A_{n'} \frac{2\pi}{\omega_a - \omega_b} \sum_{m=-\infty}^{\infty} \int dH_\perp \int dP_\theta \int dv_z$$

$$\cdot \left\{ m_j \frac{\partial f_{oj}}{\partial H_\perp} [\ell(\omega_b - \omega_j) - m(\omega_a - \omega_b)] + k \frac{\partial f_{oj}}{\partial v_z} \right\}$$

$$\cdot \frac{J_{\ell+m}(\lambda_n b) J_m(\lambda_n a) J_{\ell+m}(\lambda_{n'} b) J_m(\lambda_n a)}{\omega + m(\omega_a - \omega_b) - k v_z - \ell \omega_b} \quad . \tag{1.46}$$

Defining

$$V_x = v_x + \omega_j y \quad , \quad V_y = v_y - \omega_j x \quad , \quad V_\perp^2 = V_x^2 + V_y^2 \quad , \quad \omega_j^\pm = \omega_{\substack{a \\ b}} \quad ,$$

we may write Davidson's result for the same problem as

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_o^R drr\, J_\ell(\lambda_n r)\, J_\ell(\lambda_{n'} r) \int d^3V \frac{1}{V_\perp} \frac{\partial f_{oj}}{\partial V_\perp}$$

$$+ \frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_o^R drr \sum_{p=-\infty}^{\infty} J_\ell(\lambda_n r) J_p\left(\frac{\omega_j^- - \omega_j^-}{\omega_j^+ - \omega_j^-} \lambda_{n'} r\right) J_{\ell-p}\left(\frac{\omega_j^+ - \omega_j}{\omega_j^+ - \omega_j^-} \lambda_n r\right)$$

$$\cdot \int d^3V \sum_{m=-\infty}^{\infty} J_m^2\left(\frac{\lambda_n V_\perp}{\omega_j^+ - \omega_j^-}\right) \frac{[k(\frac{\partial}{\partial V_z} - \frac{V_z}{V_\perp}\frac{\partial}{\partial V_\perp}) + \frac{\omega - \ell\omega_j}{V_\perp}\frac{\partial}{\partial V_\perp}]f_{oj}}{\omega - \ell\omega_j^- - (p+m)(\omega_j^+ - \omega_j^-) - kV_z} \,. \qquad (1.47)$$

Note that (1.46) is somewhat simpler than (1.47) because in the derivation of (1.46) we were able to carry out analytically the integral corresponding to the r-integral in (1.47).

The two results (1.46) and (1.47) are not analytically identical. This is due to the fact that the phase space integration in (1.47) includes the paths of all particles having any part of their orbits inside R, while in (1.46) we include only those particles with orbits which lie entirely within the cylinder. However, to the extent that the plasma density at the cylinder wall is negligible, the two methods should give essentially the same numerical result.

To verify this, a numerical calculation of a particular case of the lower hybrid drift instability was carried out and the results compared to those Davidson obtains from (1.47). The equilibrium distribution function is taken to be a Gibbs distribution:

$$f_{oj} = \hat{n}_j \left(\frac{m_j}{2\pi T_j}\right)^{3/2} e^{-\frac{H_\perp - \omega_j P_\theta}{T_j}} e^{-\frac{m_j v_z^2}{2T_j}} .$$

The density profile may then be shown to have the Gaussian form

$$n_{oj}(r) = \hat{n}_j \, e^{-r^2/R_o^2} , \qquad (1.48)$$

where R_o represents the characteristic radius of the plasma column and is given by

$$R_o^2 = \frac{2(ZT_e+T_i)}{Zm_e(\omega_e\omega_{ce}-\omega_e^2)-m_i(\omega_i^2+\omega_i\omega_{ci})} . \qquad (1.49)$$

Here Z is the multiplicity of the ion charge and ω_{ce} and ω_{ci} are the absolute values of the electron and ion gyrofrequencies, respectively.

For the numerical calculation, we take k = 0, $R_c/R_o = 2.5$, $T_e/T_i = 1$, $\omega_E R_o/v_i = 3$, and $m_i m_e = 1836$, where Davidson defines ω_E as

$$R_o^2 = \frac{2(ZT_e+T_i)}{Zm_e(\omega_e\omega_{ce}-\omega_e^2)-m_i(\omega_i^2+\omega_i\omega_{ci})} \,\vdots \qquad (1.50)$$

Davidson carries out his calculation using the strongly magnetized electron and unmagnetized ion approximation, and the further assumption that $\lambda_n R_o > 1$, all of which are valid for the equilibrium configuration and modes dealt with in this calculation. In addition, Davidson shows that under these assumptions it is only necessary to consider the diagonal elements of the dispersion matrix to determine the unstable modes; to facilitate comparison of results we likewise use only the diagonal terms of (1.46).

The results of the calculation are shown in Figs. (1.4) - (1.8) for several n's and ℓ's. The circles represent the results obtained from (1.46), the squares Davidson's results. The two approaches are seen to be in quite good agreement. The discrepancies tend to be largest for large ℓ and small n; that is to say, for those modes where the potential fluctuations are localized near the conducting wall. As remarked above, the dispersion relation (1.46) does not include the effects of orbits extending outside the wall, so that strictly speaking the distribution function is no longer of the rigid rotor form. This does not affect the validity of (1.46) however, since the approach used to derive it is valid for any distribution function. The phase space integral leading to Davidson's result (1.47) however, includes orbits extending outside the cylinder, so that these particles are not treated correctly in the orbit integration. Davidson justifies this with the assumption that the plasma density at the wall is "negligible", due to the Gaussian form of the density function (1.48). As an example, if we take the density at the center of the cylinder to be $\hat{n}$, the density at the wall will be $n_o = e^{-6.25}\hat{n} = 1.93 \times 10^{-3}\hat{n}$, so that the plasma density at $r = R$ is indeed negligible compared to the density at the center. However, if we consider the mode $\ell = 30$, $n = 1$, the maximum of $\phi_1(r)$ (and thus of $n_1(r)$) occurs at approximately $r = r_m = .90R$, so that

$n_o(r_m) \cong 3.65 \times 10^{-3}\hat{n}$. Thus in this case the density at the maximum of ϕ_1 is less than twice that at the cylinder wall, and could have a significant effect on the frequency and stability of the mode. The same may be said for other high ℓ or low n modes, and this may account for much of the difference between the two sets of results as presented in Figs. (1.4) - (1.8). We note also that Davidson finds the highest growth rate to be associated with the $\ell = 44$, $n = 1$ mode, which is localized close enough to the wall to be affected by the above considerations. In fact, the difference in the two results for this mode, as shown in Fig. (1.8), amounts to about 10%, so that the effect of finite density at the wall seems to be non-negligible for the most unstable modes.

Fig 1.1

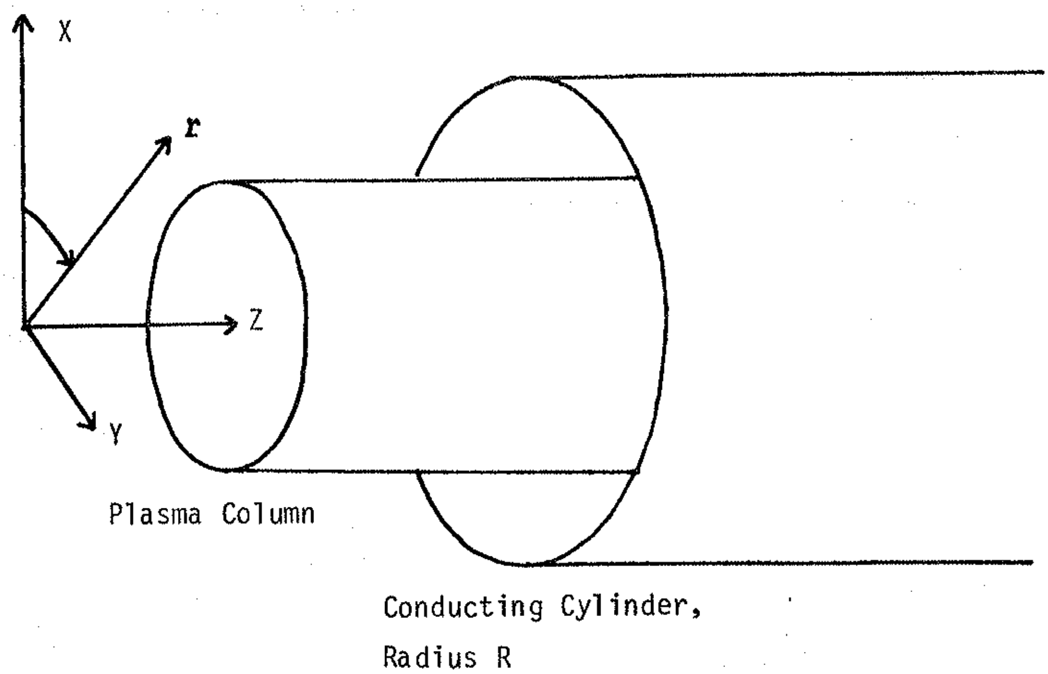
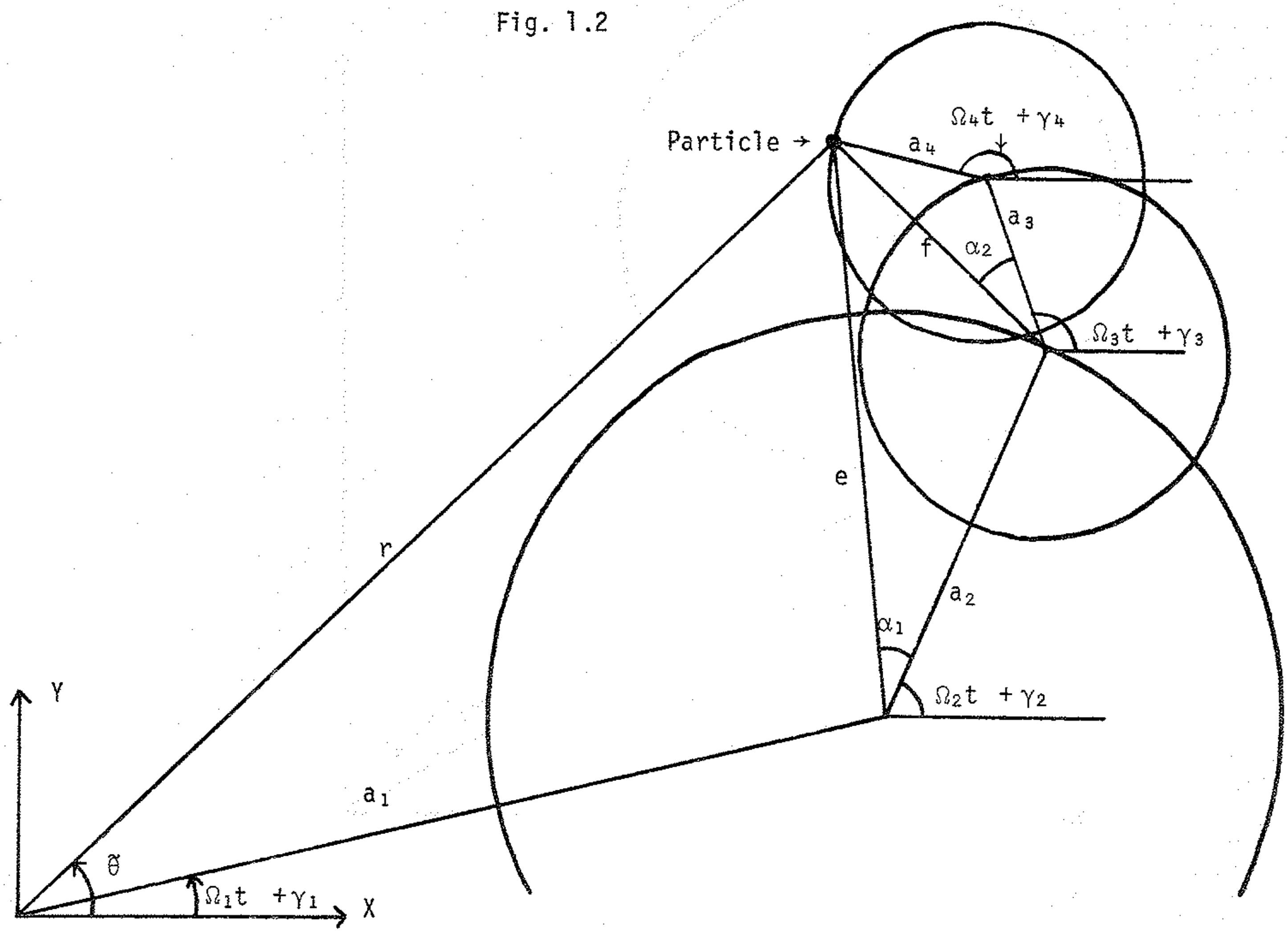
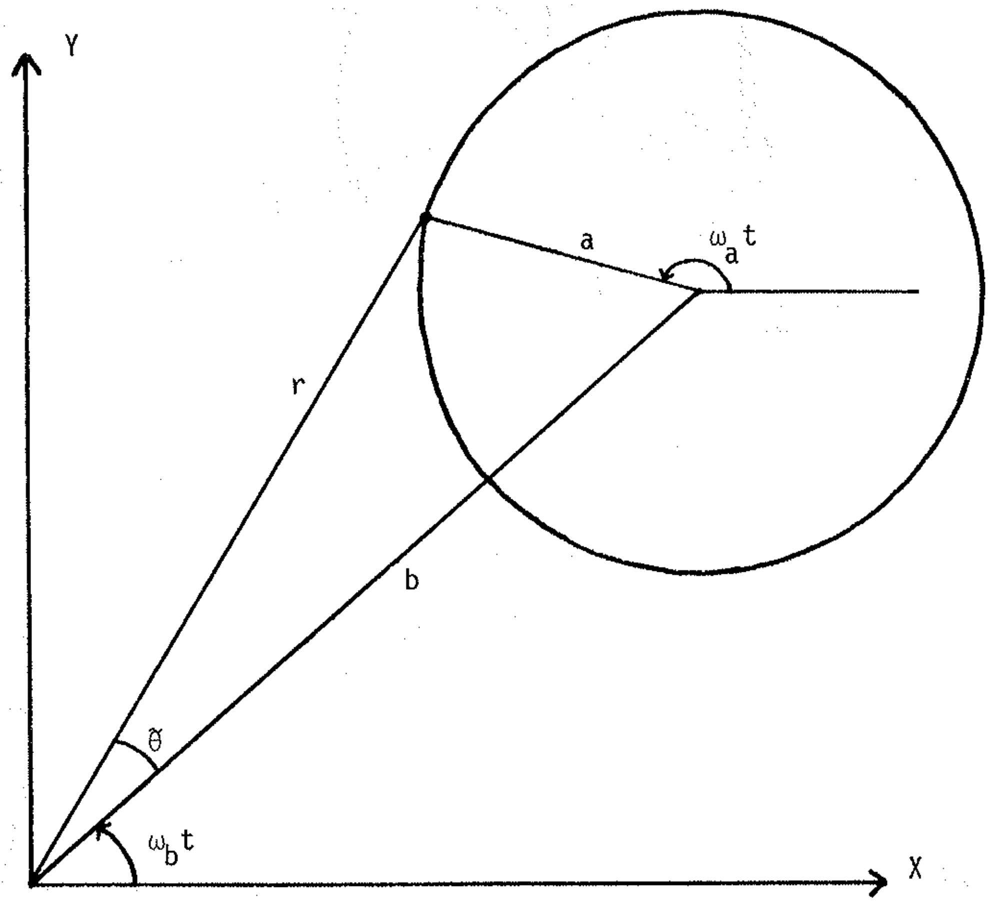


Fig. 1.2



Particle →
$\Omega_4 t + \gamma_4$
a_4
a_3
f
α_2
$\Omega_3 t + \gamma_3$
e
r
a_2
α_1
$\Omega_2 t + \gamma_2$
Y
a_1
θ
$\Omega_1 t + \gamma_1$
X

Fig. 1.3



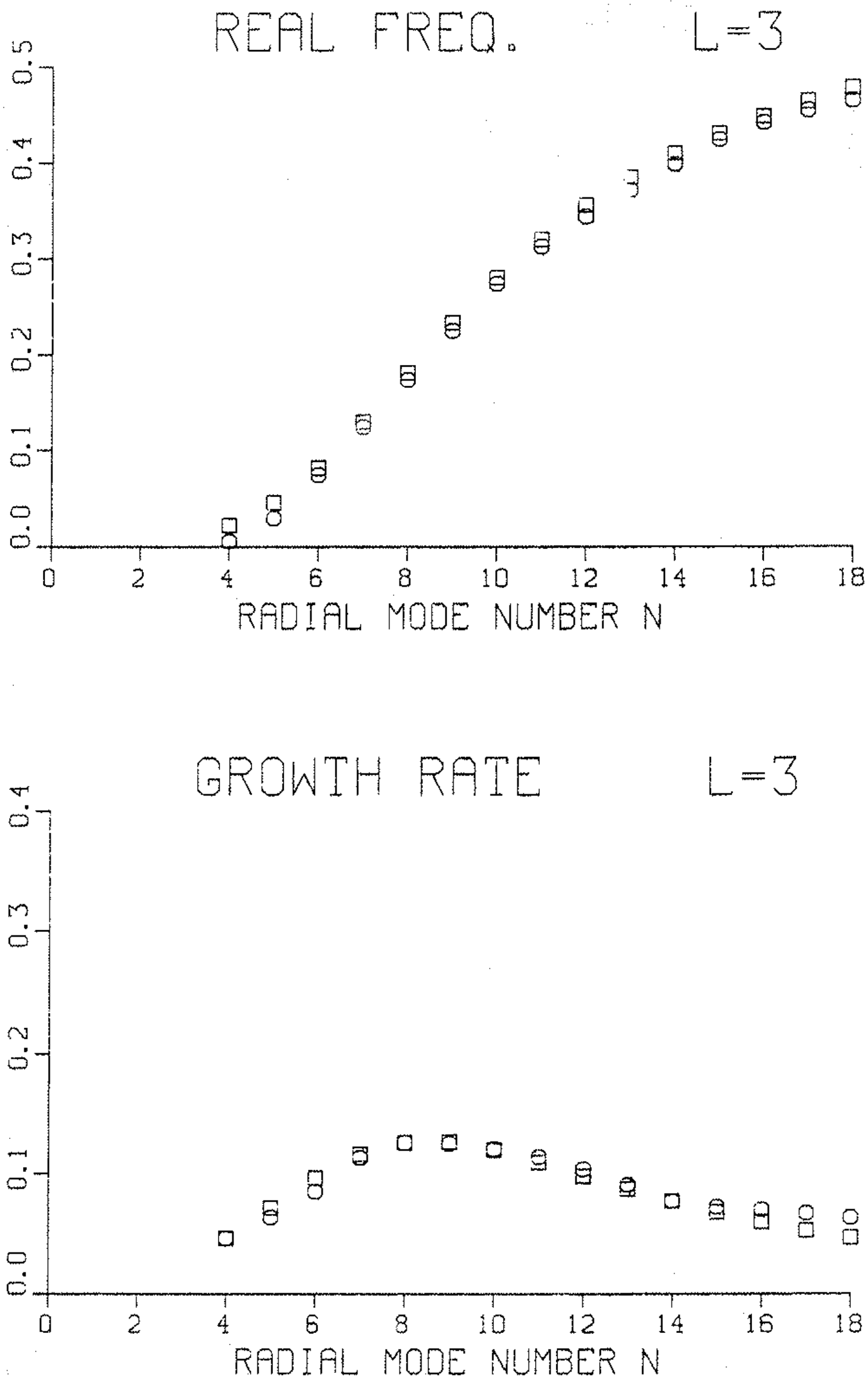
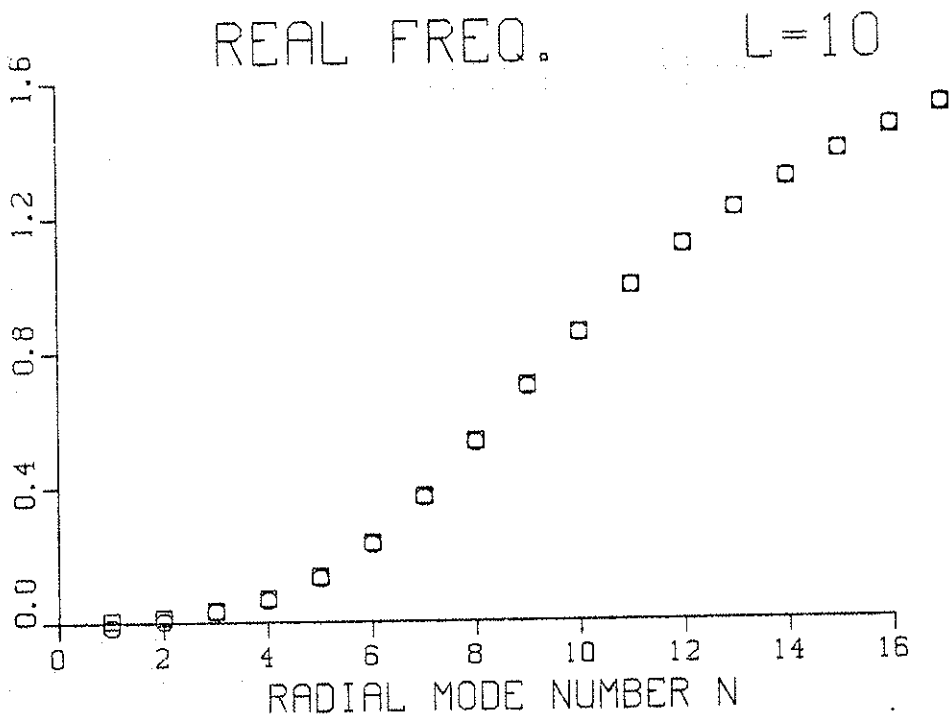

REAL FREQ. L=3
0.0
0.1
0.2
0.3
0.4
0.5
0
2
4
6
8
10
12
14
16
18
RADIAL MODE NUMBER N
GROWTH RATE L=3
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
10
12
14
16
18
RADIAL MODE NUMBER N

Fig. 1.4


REAL FREQ. L=10
1.6
1.2
0.8
0.4
0.0
0
2
4
6
8
10
12
14
16
RADIAL MODE NUMBER N

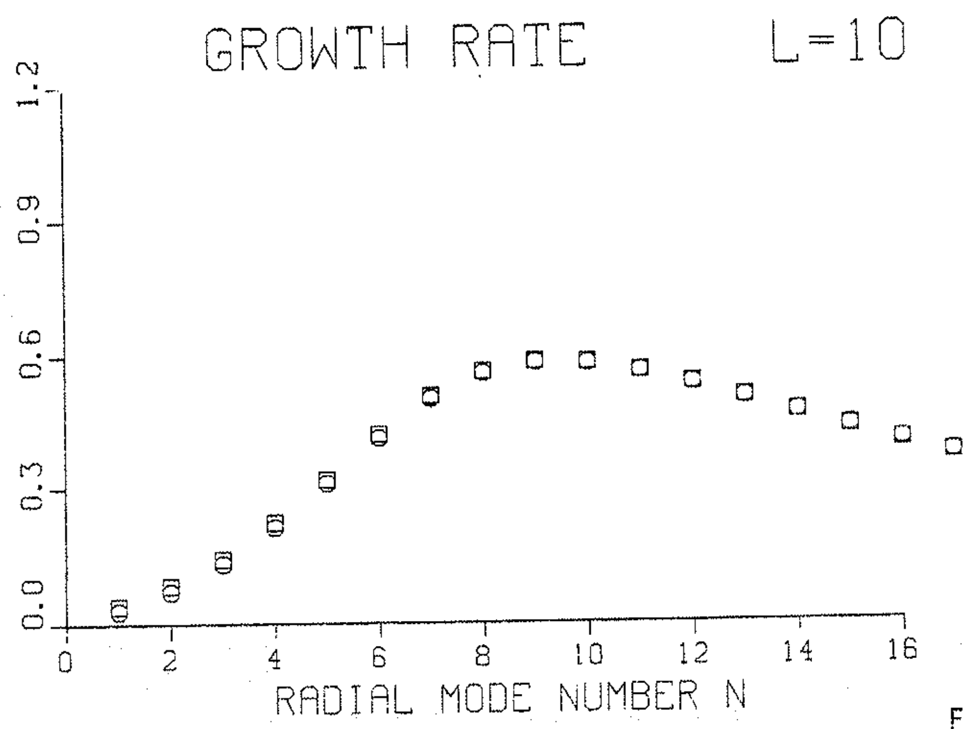

GROWTH RATE L=10
1.2
0.9
0.6
0.3
0.0
0
2
4
6
8
10
12
14
16
RADIAL MODE NUMBER N

Fig 1.5

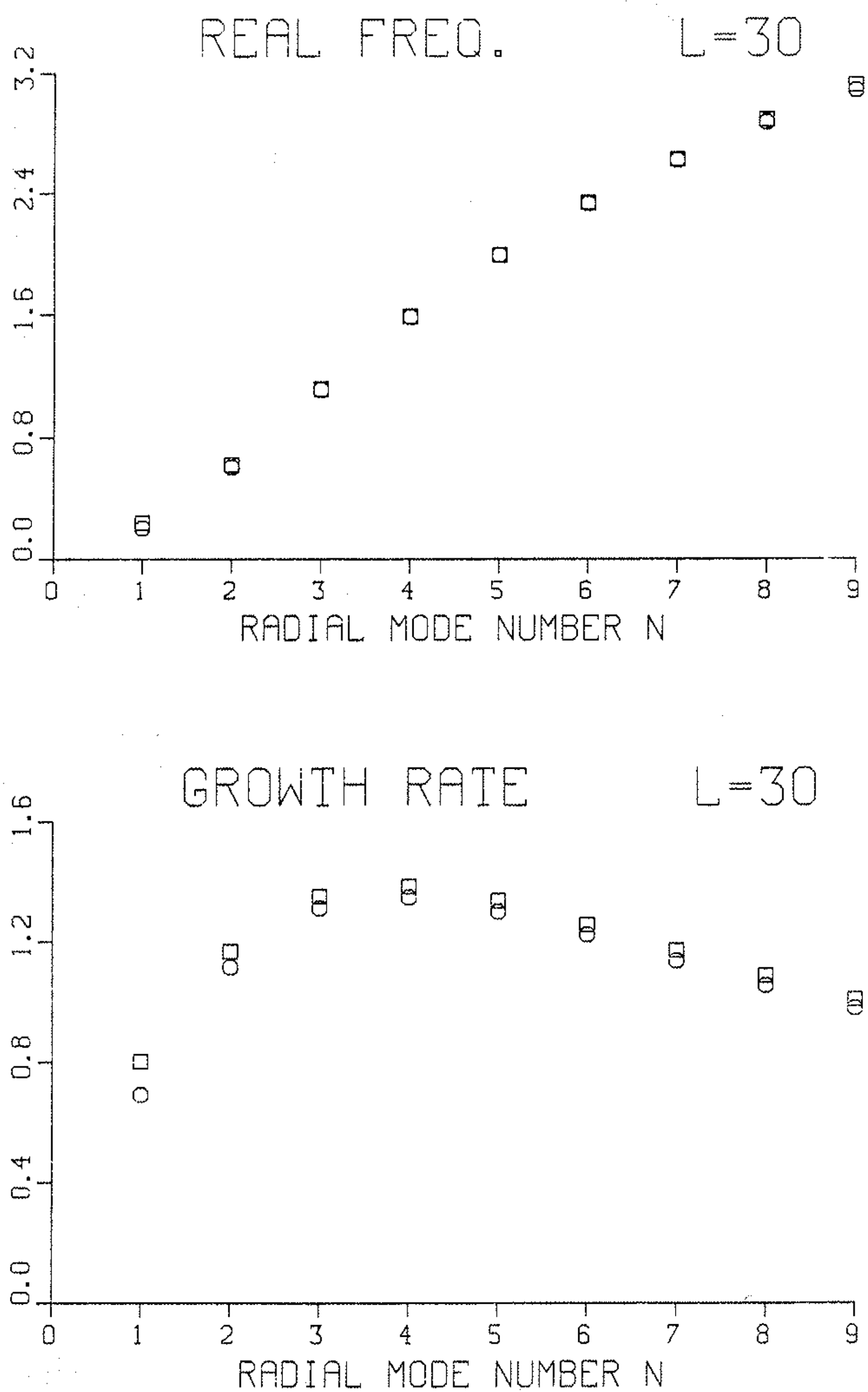

REAL FREQ.
L=30
3.2
2.4
1.6
0.8
0.0
0
1
2
3
4
5
6
7
8
9
RADIAL MODE NUMBER N
GROWTH RATE
L=30
1.6
1.2
0.8
0.4
0.0
0
1
2
3
4
5
6
7
8
9
RADIAL MODE NUMBER N

Fig 1.6

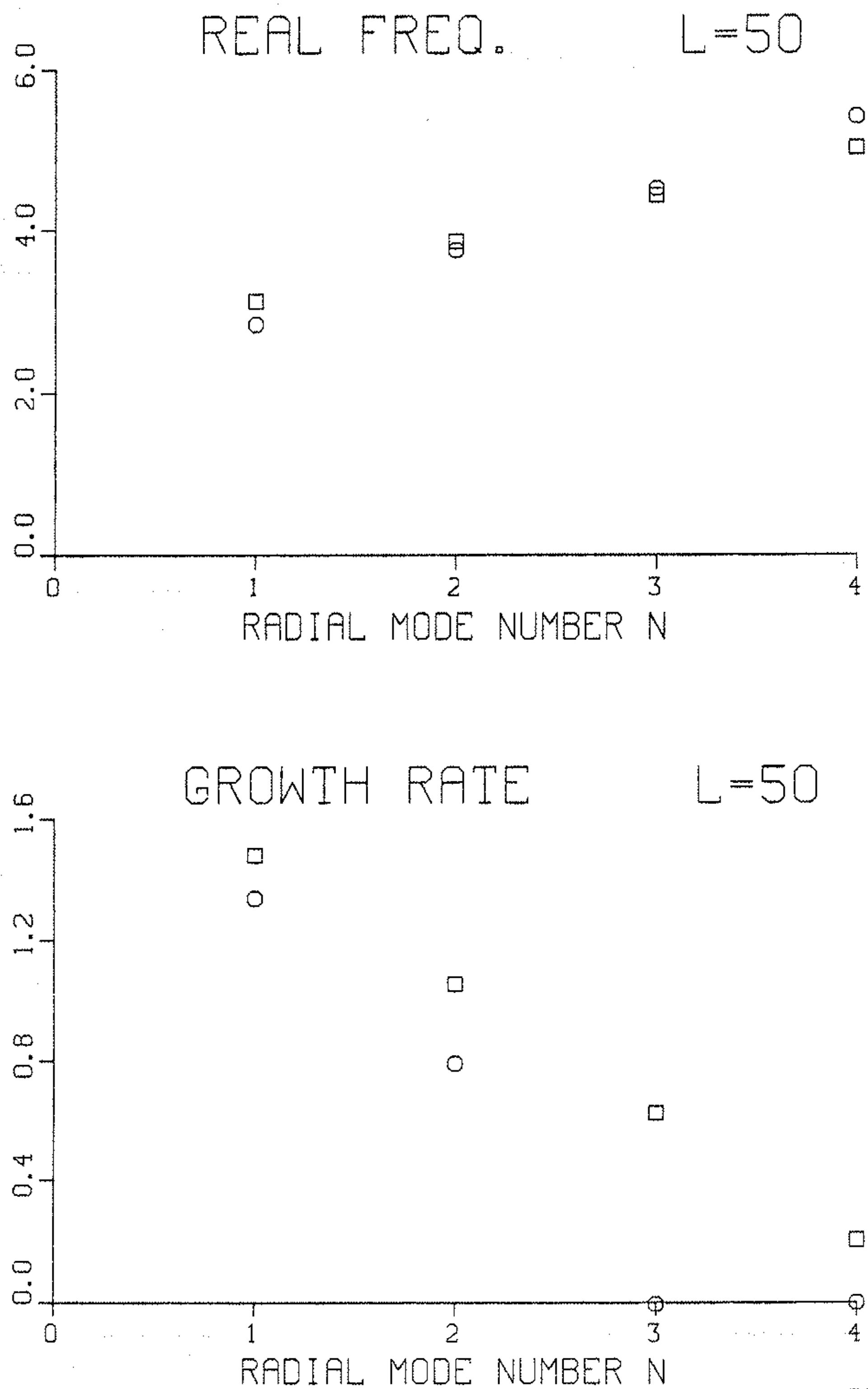

REAL FREQ. L=50
6.0
4.0
2.0
0.0
0
1
2
3
4
RADIAL MODE NUMBER N
GROWTH RATE L=50
1.6
1.2
0.8
0.4
0.0
0
1
2
3
4
RADIAL MODE NUMBER N

Fig. 1.7

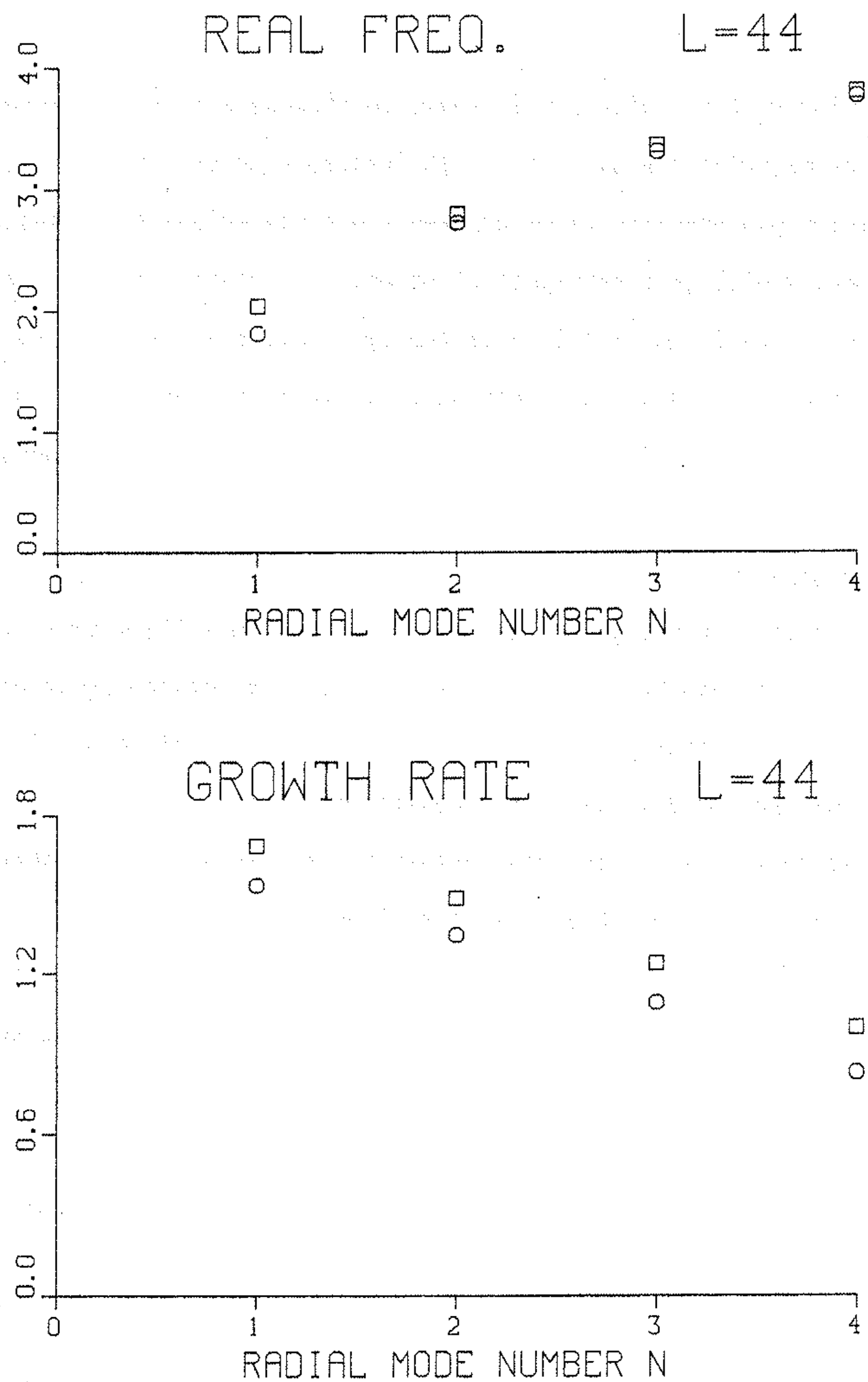

REAL FREQ.
L=44
4.0
3.0
2.0
1.0
0.0
0
1
2
3
4
RADIAL MODE NUMBER N
GROWTH RATE
L=44
1.8
1.2
0.6
0.0
0
1
2
3
4
RADIAL MODE NUMBER N

Fig. 1.8

II. EXTENSION TO ELECTROMAGNETIC CASE

For problems with $\beta \sim 1$, where considerable free energy resides in the magnetic field, or for high frequency modes with $\omega \sim ck$, we expect significant coupling between the longitudinal (electrostatic) and transverse (electromagnetic) modes. In relativistic plasmas, where a significant fraction of the particles have $v/c \cong 1$, we expect that fundamentally electromagnetic modes may be driven unstable by resonant particles. Consequently, for many problems an accurate stability analysis will require consideration of transverse as well as longitudinal modes.

In the preceding chapter we have shown how a linearized stability analysis may be carried out for the Vlasov-Poisson equations in cylindrical geometry. In this chapter we extend this analysis to the full set of Vlasov-Maxwell equations and in the following chapter we shall illustrate this formalism with two numerical analyses of relativistic beam-plasma interactions.

We again assume the geometry and coordinate system of Fig. (1.1). The linearized Vlasov-Maxwell equations in the Lorentz gauge are:

$$\left(\frac{\partial}{\partial t} + L_o\right) f_{1j}(\underline{r},\underline{v},t) =$$

$$\frac{e_j}{m_j} \left[\nabla\phi_1(\underline{r},t) - \frac{1}{c}\, \underline{v} \times (\nabla \times \underline{A}_1(\underline{r},t)) + \frac{1}{c}\frac{\partial}{\partial t}\underline{A}_1(\underline{r},t)\right] \cdot \frac{\partial f_{oj}}{\partial \underline{v}} \quad , \tag{2.1}$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\underline{A}_1(\underline{r},t) = -\sum_j \frac{4\pi e_j}{c}\int d^3v\underline{v}f_{1j}(\underline{r},\underline{v},t)\ ,$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\phi_1(\underline{r},t) = -\sum_j 4\pi e_j\int d^3v f_{1j}(\underline{r},\underline{v},t)\ , \qquad (2.2)$$

with gauge condition

$$\nabla \cdot \underline{A}_1 + \frac{1}{c}\frac{\partial\phi_1}{\partial t} = 0\ . \qquad (2.3)$$

We assume that all perturbed quantities have $e^{-i\omega t}$ time dependence, with $\text{Im}(\omega) > 0$.

The Lorentz condition (2.3) does not uniquely specify $\underline{A}$, as we may introduce a further restricted gauge transformation[8]

$$\underline{A} \rightarrow \underline{A} + \nabla\Lambda\ ,$$

$$\phi \rightarrow \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} = \phi + \frac{i\omega}{c}\Lambda\ , \qquad (2.4)$$

where $\Lambda(r,\theta,z,t)$ is any function satisfying

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\Lambda = (\nabla^2 + \frac{\omega^2}{c^2})\Lambda = 0\ . \qquad (2.5)$$

We now show that we can use the transformation (2.4) to pick a gauge in which the potentials satisfy the boundary conditions

$$\nabla \cdot \underline{A}_1\Big|_{r=R} = \phi_1\Big|_{r=R} = 0\ , \qquad (2.6)$$

where R is again the radius of the conducting cylinder.

Since the potentials are periodic in the θ and z coordinates, we may write

$$\phi(R,\theta,z) = \sum_{\ell,k} \Phi(\ell,k)\ e^{i(\ell\theta+kz)} \ . \tag{2.7}$$

If we define $\Lambda(r,\theta,z)$ by

$$\Lambda(r,\theta,z) = \sum_{\ell,k} \Lambda(\ell,k)\ J_\ell(\lambda_{\ell,k} r)\ e^{i(\ell\theta+kz)} \ , \tag{2.8}$$

where $\Lambda(\ell,k)$ and $\lambda_{\ell,k}$ are constants depending on ℓ and k, then we have

$$(\nabla^2 + \frac{\omega^2}{c^2})\Lambda = \sum_{\ell,k} (\frac{\omega^2}{c^2} - \lambda_{\ell,k}^2 - k^2)\ J_\ell(\lambda_{\ell,k} r)\ e^{i(\ell\theta+kz)} \ . \tag{2.9}$$

Clearly Λ will satisfy (2.9) if we choose

$$\lambda_{\ell,k} = \sqrt{\frac{\omega^2}{c^2} - k^2} \ , \tag{2.10}$$

and from (2.7) and (2.8) we see that (2.6) will be satisfied if

$$\Lambda(\ell,k) = \Phi(\ell,k)/\ \frac{i\omega}{c}\ J_\ell(\lambda_{\ell,k} R) \ . \tag{2.11}$$

Note that since $\mathrm{Im}(\omega) > 0$, (2.10) shows that $\lambda_{\ell,k}$ is always complex (non-real), and since all the zeros of the Bessel functions of the first kind are real, the denominator in (2.11) never vanishes. Consequently, we may always impose the boundary condition (2.6), and in doing so uniquely determine the gauge of $\underline{A}$ and ϕ (up to an additive constant in $\underline{A}$).

Using the gauge condition (2.5) we may eliminate ϕ from the equations:

$$\phi = -\frac{ic}{\omega}\nabla\cdot\underline{A}\ , \tag{2.12}$$

where we now drop the subscript "ℓ" on the perturbed quantities. E_θ, E_z, and B_r must be zero at the cylinder wall, so from (2.6), (2.12), and

$$\underline{E} = -\nabla\phi + \frac{i\omega}{c}\underline{A}$$

we have for boundary conditions on $\underline{A}$:

$$\nabla\cdot\underline{A}\Big|_{r=R} = 0\quad ,\quad A_\theta\Big|_{r=R} = 0\quad ,\quad A_z\Big|_{r=R} = 0\ . \tag{2.13}$$

In the preceding chapter, we expanded the scalar potential in cylindrical harmonics in order to obtain a matrix dispersion relation. These expansion functions were chosen to satisfy the boundary condition $\phi(r) = 0$. For the electromagnetic case we now define

$$A^{\pm} = A_x \pm iA_y\ ,$$

and show that a suitable expansion for the vector potential satisfying the boundary conditions (2.13) is the following:

$$A^+(\underline{r}) = \sum_{\ell,n,k} [\alpha_{n\ell k} J_{\ell+1}(\lambda'_{n\ell} r) + \beta_{n\ell k} J_{\ell+1}(\lambda_{n\ell} r)] e^{i[(\ell+1)\theta + kz]} ,$$

$$A^-(\underline{r}) = \sum_{\ell,n,k} [\alpha_{n\ell k} J_{\ell-1}(\lambda'_{n\ell} r) - \beta_{n\ell k} J_{\ell-1}(\lambda_{n\ell} r)] e^{i[(\ell-1)\theta + kz]} ,$$

$$A_z(\underline{r}) = \sum_{\ell,n,k} \gamma_{n\ell k} J_\ell(\lambda_{n\ell} r) e^{i(\ell\theta + kz)} . \tag{2.14}$$

Here $\alpha_{n\ell k}$, $\beta_{n\ell k}$, $\gamma_{n\ell k}$ are the expansion coefficients representing the three independent components of the potentials, $\lambda_{n\ell}$ is the n^{th} root of $J_\ell(R) = 0$ and $\lambda'_{n\ell}$ is the n^{th} root of $J_\ell(R) = 0$. Thus the third equation in (2.14) shows that the third boundary condition in (2.13) is satisfied. To show that the second boundary condition is satisfied, we must express A_θ in terms of A^+ and A^-. We have

$$A_\theta = -A_x \sin\theta + A_y \cos\theta \quad , \quad A_x = \frac{1}{2} (A^+ + A^-) \quad , \quad A_y = \frac{1}{2i} (A^+ - A^-) ,$$

and thus

$$A_\theta = -\frac{1}{2} (A^+ + A^-) \sin\theta - \frac{i}{2} (A^+ - A^-) \cos\theta = \frac{1}{2} [A^- e^{i\theta} - A^+ e^{-i\theta}] .$$

Using (2.14) this gives

$$A_\theta = \frac{i}{2} \sum_{\ell,n,k} \{\alpha_{n\ell k} [J_{\ell=1}(\lambda'_{n\ell} r) - J_{\ell+1}(\lambda'_{n\ell} r)]$$

$$- \beta_{n\ell k} [J_{\ell-1}(\lambda_{n\ell} r) + J_{\ell+1}(\lambda_{n\ell} r)]\} e^{i(\ell\theta + kz)} .$$

Using the Bessel function identities

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad , \quad J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

we then have

$$A_\theta = i \sum_{\ell,n,k} \{\alpha_{n\ell k} [J_\ell'(\lambda_{n\ell}' r) - \beta_{n\ell k} \cdot \frac{\ell}{\lambda_{n\ell} r} J_\ell(\lambda_{n\ell} r)] \, e^{i(\ell\theta+kz)}\} \ .$$

Since

$$J_\ell(\lambda_{n\ell} r) = J_\ell'(\lambda_{n\ell}' R) = 0 \ ,$$

we see that the second of the boundary conditions (2.13) is satisfied.

From a similar though longer calculation (given in Appendix B), we find:

$$\nabla \cdot \underline{A} = \sum_{\ell,n,k} (\beta_{n\ell k} \, \lambda_{n\ell} + ik_z \, \gamma_{n\ell k}) \, J_\ell(\lambda_{n\ell} r) \, e^{i(\ell\theta+kz)}, \qquad (2.15)$$

and thus the first of (2.13) is also satisfied.

As in the electrostatic case, we now determine the perturbed distribution functions by integration over unperturbed orbits. From (2.1) we obtain:

$$f_{1j}(\underline{r},\underline{v}) = \frac{e_j}{m_j} \int_{-\infty}^{t} dt' \, e^{i\omega(t-t')} [\nabla\phi(\underline{r}) - \frac{i\omega}{c} \underline{A}(r) - \frac{\underline{v}\times(\nabla\times\underline{A}(\underline{r}))}{c}] \cdot \frac{\partial f_{oj}}{\partial \underline{v}} \ . \qquad (2.16)$$

Using

$$\frac{\partial f_{oj}}{\partial \underline{v}} = \frac{\partial f_{oj}}{\partial H} m_j \underline{v} + \frac{\partial f_{oj}}{\partial P_\theta} m_j r\theta + \frac{\partial f_{oj}}{\partial P_z} m_j \hat{z} \tag{2.17}$$

and substituting (2.14) into (2.16), we obtain after considerable algebra:

$$f_{1j} = e_j \int_{-\infty}^{t} dt' \, e^{i\omega(t-t')} \sum_{\ell,n,k_z} \left\{ \frac{\partial f_{oj}}{\partial H} \left(-\frac{ic}{\omega}\right)\left[(\lambda_{n\ell}\beta_{n\ell k} + ik_z\gamma_{n\ell k}) \frac{d}{dt'} C_{n\ell} \right.\right.$$

$$+ \frac{1}{2} \alpha_{n\ell k}(C'_{n\ell+1} v'_- + C'_{n\ell-1} v'_+) + \frac{1}{2} \beta_{n\ell k}(C_{n\ell+1} v'_- - C_{n\ell-1} v'_+)\Big]$$

$$+ \frac{\partial f_{oj}}{\partial P_\theta} \Big[\frac{\ell c}{\omega} (\lambda_{n\ell}\beta_{n\ell k} + ik_z\gamma_{n\ell k}) \overline{C}_{n\ell} - \frac{i\ell v'_z}{c} \gamma_{n\ell k} \overline{C}_{n\ell}$$

$$- \frac{1}{2} \frac{k_z v'_z}{c} \alpha_{n\ell k}(r'_+ C'_{n\ell-1} - r'_- C'_{n\ell+1}) + \frac{\ell k_z v'_z}{\lambda_{n\ell} c} \beta_{n\ell k} \overline{C}_{n\ell}$$

$$+ \frac{\omega\alpha_{n\ell k}}{2c} (C'_{n\ell-1} r'_+ - C'_{n\ell+1} r'_-) \tag{2.18}$$

$$- \frac{\omega\beta_{n\ell k}}{2c} (C_{n\ell-1} r'_+ + C_{n\ell+1} r'_-) - \frac{i\lambda'_{n\ell}}{2c} (v'_+ r'_- + v'_- r'_+) \alpha_{n\ell k} \overline{C}'_{n\ell}$$

$$+ \frac{\partial f_{oj}}{\partial P_z} \Big[\frac{k_z c}{\omega} (\lambda_{n\ell}\beta_{n\ell k} + ik_z\gamma_{n\ell k}) \overline{C}_{n\ell} - \frac{\lambda_{n\ell}}{2c} \gamma_{n\ell k}$$

$$\cdot (v'_- C_{n\ell+1} - v'_+ C_{n\ell-1}) + \gamma_{n\ell k} \overline{C}_{n\ell}$$

$$\left. - \frac{ik_z}{2c} \alpha_{n\ell k}(v'_+ C'_{n\ell-1} + v'_- C'_{n\ell+1}) - \frac{ik_z}{2c} \beta_{n\ell k}(v'_- C_{n\ell+1} - v'_+ C_{n\ell-1})\Big]\right\} .$$

In (2.18)

$$v'_{\pm} = v'_x(t') \pm iv'_y(t') \quad , \quad r'_{\pm} = x'(t') \pm iy'(t') \; .$$

Here the prime indicates dependence on t' and

$$\underline{r}'(t'=t) = \underline{r} \quad , \quad \underline{v}'(t'=t) = \underline{v} \; ,$$

and we have defined

$$\overline{C}_{n\ell} = J_\ell(\lambda_{n\ell}r')e^{i(\ell\theta'+k_z z')} \; , \; C_{n\ell\pm1} = J_{\ell\pm1}(\lambda_{n\ell}r')e^{i[(\ell\pm1)\theta'+k_z z']} \; ,$$

$$\overline{C}'_{n\ell} = J_\ell(\lambda'_{n\ell}r')e^{i(\ell\theta'+k_z z')} \; , \; C'_{n\ell\pm1} = J_{\ell\pm1}(\lambda'_{n\ell}r')e^{i[(\ell\pm1)\theta'+k_z z']} \; .$$

Now we can use (1.24), (1.25) and (1.28) to express $e^{ik_z z'}$, $r'_{\pm}$, $v'_{\pm}$, and the C's as Fourier series in time and proceed with the integration as in the electrostatic case, obtaining a set of linear equations in the expansion coefficients $\alpha_{n\ell k}$, $\beta_{n\ell k}$, $\gamma_{n\ell k}$. This is quite cumbersome for the general equation (2.18), however, so for the purpose of illustration and with a view toward the applications in the next chapter we carry out the calculation in full only for the simple case of $k = A_z = 0$, i.e., we consider only perturbations depending solely on r and θ. We note, however, that the extension to z-dependent perturbations, though algebraically lengthy, is straightforward.

We therefore take $\gamma_{n\ell k} = k = 0$, so that Eq. (2.18) becomes

$$f_{1j} = e_j \int_{-\infty}^{t} dt' \; e^{i\omega(t-t')} \sum_{\ell,n} \Bigg[\frac{\partial f_{oj}}{\partial H} \Big\{ - \frac{ic}{\omega} \beta_{n\ell} \lambda_{n\ell} \frac{d}{dt'} [J_\ell(\lambda_{n\ell} r') e^{i\ell\theta'}]$$

$$- \frac{i\omega}{2c} \alpha_{n\ell} [J_{\ell+1}(\lambda'_{n\ell} r') v'_- \; e^{i(\ell+1)\theta'} + J_{\ell-1}(\lambda'_{n\ell} r') \; v'_+ \; e^{i(\ell-1)\theta'}]$$

$$- \frac{i\omega}{2c} \lambda_{n\ell} [J_{\ell+1}(\lambda_{n\ell} r') v'_- \; e^{i(\ell+1)\theta'} - J_{\ell-1}(\lambda_{n\ell} r') v'_+ \; e^{i(\ell-1)\theta'}] \Big\}$$

$$+ \frac{\partial f_{oj}}{\partial P_\theta} \Big\{ \frac{\ell c}{\omega} \lambda_{n\ell} \lambda_{n\ell} \; J_\ell(\lambda_{n\ell} r') e^{i\ell\theta'} \tag{2.19}$$

$$- \frac{i}{2c} (v'_+ r'_- + v'_- r'_+) \; \alpha_{n\ell} \lambda'_{n\ell} \; J_\ell(\lambda'_{n\ell} r') e^{i\ell\theta'} + \frac{\omega}{2c} \alpha_{n\ell}$$

$$\cdot \; [J_{\ell-1}(\lambda'_{n\ell} r') r'_+ \; e^{i(\ell-1)\theta'} - J_{\ell+1}(\lambda'_n \; r') r'_- e^{i(\ell+1)\theta'}]$$

$$- \frac{\omega}{2c} \beta_{n\ell} [J_{\ell-1}(\lambda_n \; r') r'_+ \; e^{i(\ell-1)\theta'} + J_{\ell+1}(\lambda_{n\ell} r') r'_- \; e^{i(\ell+1)\theta'}] \Big\} \Bigg] \; .$$

Next we use the techniques of the previous chapter to represent the integrand of (2.19) as a Fourier series in time. Since we still have only one non-ignorable coordinate, the arguments which in the electrostatic case led to the expansions (1.25) and (1.28) remain valid, and terms of the form $r'_\pm$, $v'_\pm$, and $J_\ell(r') e^{i\ell\theta'}$ may all be readily represented as Fourier series in time. To make the method as clear as possible and avoid unduly lengthy algebra, we utilize only the first two terms in the expansions (1.25) and (1.28) in the following calculations. (The extension to more than two terms is again straightforward, as in the electrostatic case.) Two terms will also be found to be sufficient for the numerical calculations of the next chapter.

Thus as in Chapter I we may write:

$$r'_{\pm} = r'e^{\pm i\theta'} = ae^{\pm i\omega_a t'} + be^{\pm i\omega_b t'} \ , \ v'_{\pm} = \pm i\omega_a ae^{\pm i\omega_a t'} \pm i\omega_b be^{\pm i\omega_b t'} \ ,$$

$$J_\ell(\lambda r')e^{i\ell\theta'} = e^{i\ell\omega_b t'} \sum_{m=-\infty}^{\infty} J_{\ell+m}(\beta b)\ J_m(\lambda a)(-1)^m\ e^{-im(\omega_a-\omega_b)t'} \ , \tag{2.20}$$

where here ω_a and ω_b may depend on the constants of the motion H and P_θ.

From Appendix B we have the results

$$\nabla \cdot \underline{A} = \sum_{\ell,n} \beta_{n\ell}\ \lambda_{n\ell}\ J_\ell(\lambda_{n\ell} r)\ e^{i\ell\theta} \ ,$$

$$(\nabla \times \underline{A})_z = -i \sum_{\ell,n} \alpha_{n\ell}\ \lambda'_{n\ell}\ J_\ell(\lambda'_{n\ell} r)\ e^{i\ell\theta} \ .$$

Equation (2.2) then yields

$$\begin{aligned}(\nabla^2 + \frac{\omega^2}{c^2})\ \nabla \cdot \underline{A} &= \sum_{\ell,n} (-\lambda^2_{n\ell} + \frac{\omega^2}{c^2})\ \beta_{n\ell}\ \lambda_{n\ell}\ J_\ell(\lambda_{n\ell} r)\ e^{i\ell\theta} \\ &= -i\omega \sum_j \frac{4\pi e_j}{c} \int d^3v\ f_{1j} \ ,\end{aligned} \tag{2.21}$$

$$\begin{aligned}(\nabla^2 + \frac{\omega^2}{c^2})(\nabla\times\underline{A})_z &= -i \sum_{\ell,n} (-\lambda'^2_{n\ell} + \frac{\omega^2}{c^2})\ \alpha_{n\ell}\ \lambda'_{n\ell}\ J_\ell(\lambda'_{n\ell} r)\ e^{i\ell\theta} \\ &= - \sum_j \frac{4\pi e_j}{c} \int d^3v\ [\nabla \times (\underline{v} f_{1j})] \ .\end{aligned} \tag{2.22}$$

To isolate one term on the left side of (2.21) and (2.22), we multiply (2.21) by r $J_\ell(\lambda_{n'\ell} r)\ e^{-i\ell\theta}$ and (2.22) by r $J_\ell(\lambda'_{n'r} r)\ e^{-i\ell\theta}$ and integrate over r and θ to obtain:

$$(-\lambda^2_{n'\ell} + \frac{\omega^2}{c^2})[\frac{R^2}{2} J_{\ell+1}(\lambda_{n\ell}R)]\ \lambda_{n\ell}\ \delta_{nn'}\ \beta_{n\ell} \tag{2.23}$$

$$= -\omega \sum_j \frac{4\pi e_j}{c} \int drr\ J_\ell(\lambda_{n'\ell} r)\ e^{-i\ell\theta} \int d^2v\ f_{1j}\ ,$$

$$-i\ \frac{R^2}{2}\ [1-(\frac{\ell}{\lambda'_{n'\ell}R})^2]\ J^2_\ell(\lambda'_{n'\ell}R)(-\lambda'^2_{n'\ell} + \frac{\omega^2}{c^2})\ \lambda'_{n'\ell}\ \delta_{nn'}\ \alpha_{n\ell} \tag{2.24}$$

$$= - \sum_j \frac{4\pi e_j}{c} \int drr\ J_\ell(\lambda'_{n'\ell} r)\ e^{-i\ell\theta} \int d^2v[\nabla\times(\underline{v}f_{1j})]\ .$$

Note that the right sides of (2.23) and (2.24) do not depend on θ since f_{1j} contains $e^{i\ell\theta}$; see the discussion preceding Eq. (1.16). Also, we have taken the integral over d^2v rather than d^3v since v_z does not enter the calculations. If f_o depends on v_z, we take the f_1 in (2.23) and (2.24) to have already been integrated over v_z. Next we must substitute (2.20) into (2.19) to obtain f_{1j}:

$$f_{1j} = e_j \int_{-\infty}^{t} dt'\ e^{i\omega(t-t')} \sum_{n,m} (-1)^m\ [\frac{\partial f_{oj}}{\partial H_\perp}\ \{X\} + \frac{\partial f_{oj}}{\partial P_\theta}\ \{Y\}]\ ,$$

where

$$\{X\} = \{-\frac{ic}{\omega}\beta_{n\ell}\lambda_{n\ell}\frac{d}{dt'}[J_{\ell+m}(\lambda_{n\ell}b)\,J_m(\lambda_{n\ell}a)\,e^{i[\ell\omega_b-m(\omega_a-\omega_b)]t'}]$$

$$-\frac{i\omega}{2c}[(-i\omega_a a\,e^{-i\omega_a t'} - i\omega_b b\,e^{-i\omega_b t'})\{\alpha_{n\ell}\,J_{\ell+m+1}(\lambda'_{n\ell}b)\,J_m(\lambda'_{n\ell}a)$$

$$+\beta_{n\ell}\,J_{\ell+m+1}(\lambda_{n\ell}b)\,J_m(\lambda_{n\ell}a)\}\cdot e^{i[(\ell+1)\omega_b-m(\omega_a-\omega_b)]t'}$$

$$+(i\omega_a a\,e^{i\omega_a t'} + i\omega_b b\,e^{i\omega_b t'})\{\alpha_{n\ell}\,J_{\ell+m-1}(\lambda'_{n\ell}b)\,J_m(\lambda'_{n\ell}a)$$

$$-\beta_{n\ell}\,J_{\ell+m-1}(\lambda_{n\ell}b)\,J_m(\lambda_{n\ell}a)\}\cdot e^{i[(\ell-1)\omega_b-m(\omega_a-\omega_b)]t'}]\},$$

$$\{Y\} = \{\frac{\ell c}{\omega}\beta_{n\ell}\lambda_n\;J_{\ell+m}(\lambda_{n\ell}b)\,J_m(\lambda_{n\ell}a)\,e^{-im(\omega_a-\omega_b)t'}$$

$$+\frac{\omega}{2c}[(ae^{i\omega_a t'} + be^{i\omega_b t'})\{\alpha_{n\ell}\,J_{\ell+m-1}(\lambda'_{n\ell}b)\,J_m(\lambda'_{n\ell}a)$$

$$-\beta_{n\ell}\,J_{\ell+m-1}(\lambda_{n\ell}b)\,J_m(\lambda_{n\ell}a)\}\cdot e^{i[(\ell-1)\omega_b-m(\omega_a-\omega_b)]t'}$$

$$-(ae^{-i\omega_a t'} + be^{-i\omega_b t'})\{\alpha_{n\ell}\,J_{\ell+m+1}(\lambda'_{n\ell}b)\,J_m(\lambda'_{n\ell}a)$$

$$+\beta_{n\ell}\,J_{\ell+m-1}(\lambda_{n\ell}b)\,J_m(\lambda_{n\ell}a)\}\cdot e^{i[(\ell+1)\omega_b-m(\omega_a-\omega_b)]t'}$$

$$+\frac{ab}{2c}(\omega_a-\omega_b)[e^{i(\omega_a-\omega_b)t'} - e^{-i(\omega_a-\omega_b)t'}]\,\alpha_{n\ell}\lambda'_{n\ell}\,J_{\ell+m}(\lambda'_{n\ell}b)$$

$$\cdot\,J_m(\lambda'_{n\ell}a)\cdot e^{i[\ell\omega_b-m(\omega_a-\omega_b)]t'}]\}\,.$$

After performing the time integral we have

$$f_{1j} = e_j \sum_{n,m} (-1)^m \left[\frac{\partial f_{oj}}{\partial H_\perp} \{U\} + \frac{\partial f_{oj}}{\partial P_\theta} \{V\}\right] \frac{e^{i[\ell\omega_b - m(\omega_a-\omega_b)]t}}{i[\ell\omega_b - m(\omega_a-\omega_b)-\omega]} , \quad (2.25)$$

where

$$\{U\} = \Bigg\{\frac{[\ell\omega_b - m(\omega_a-\omega_b)]c}{\omega} \beta_{n\ell}\lambda_{n\ell} J_{\ell+m}(\lambda_{n\ell}b) J_m(\lambda_{n\ell}a) \quad (2.26)$$

$$+ \frac{\omega\omega_c a}{c} [\alpha_{n\ell} J_{\ell+m}(\lambda'_{n\ell}b) J'_m(\lambda'_{n\ell}a) + \beta_{n\ell} \frac{m}{\lambda_{n\ell}a} J_{\ell+m}(\lambda_{n\ell}b) J_m(\lambda_{n\ell}a)]$$

$$+ \frac{\omega\omega_b b}{c} [\alpha_{n\ell} J'_{\ell+m}(\lambda'_{n\ell}b) J_m(\lambda'_{n\ell}a) - \beta_{n\ell} \frac{\ell+m}{\lambda_{n\ell}b} J_{\ell+m}(\lambda'_{n\ell}b) J_m(\lambda_{n\ell}a)]\Bigg\},$$

$$\{V\} = \Bigg\{\frac{\ell c}{\omega} \beta_{n\ell}\lambda_{n\ell} J_{\ell+m}(\lambda_{n\ell}b) J_m(\lambda_{n\ell}a) \quad (2.27)$$

$$+ \frac{\omega}{c} [a\alpha_{n\ell} J_{\ell+m}(\lambda'_{n\ell}b) J'_m(\lambda'_{n\ell}a) + a\beta_{n\ell} \frac{m}{\lambda_{n\ell}a} J_{\ell+m}(\lambda_{n\ell}b) J_m(\lambda_{n\ell}a)$$

$$+ b\alpha_{n\ell} J'_{\ell+m}(\lambda'_{n\ell}b) J_m(\lambda'_{n\ell}a) - b\beta_{n\ell} \frac{\ell+m}{\lambda_{n\ell}b} J_{\ell+m}(\lambda_{n\ell}b) J_m(\lambda_{n\ell}a)]$$

$$+ \frac{ab}{c} (\omega_a-\omega_b) \alpha_{n\ell}\lambda'_{n\ell} [\frac{m}{\lambda'_{n\ell}a} J'_{\ell+m}(\lambda'_{n\ell}b) J_m(\lambda'_{n\ell}a)$$

$$+ \frac{\ell+m}{\lambda'_{n\ell}b} J_{\ell+m}(\lambda'_{n\ell}b) J'_m(\lambda'_{n\ell}a)]\Bigg\} .$$

Using (1.36), (1.30) and (1.31), we change the variables of integration in (2.23) from r and $\underline{v}$ to t, H and P_θ and get

$$(-\lambda_{n\ell}^2 + \frac{\omega^2}{c^2})\ \beta_{n\ell}\lambda_{n\ell}\ [\frac{R^2}{2}\ J_{\ell+1}^2(\lambda_{n\ell}R)]$$

$$= -i\omega \sum_j \frac{4\pi e_j}{m_j^2 c} \int dH\ dP_\theta dt \sum_m (-1)^m\ J_{\ell+m}(\lambda_{n\ell}b)$$

$$\cdot\ J_m(\lambda_{n\ell}a)\ e^{i[m(\omega_a-\omega_b)-\ell\omega_b]t}\ f_{1j}\ . \qquad (2.28)$$

Combining (2.25) and (2.28) we can do the integral over t and get

$$(-\lambda_n^2 + \frac{\omega^2}{c^2})\ \beta_{n\ell}\lambda_{n\ell}\ [\frac{R^2}{2}\ J_{\ell+1}^2(\lambda_{n\ell}R)]$$

$$= -\omega \sum_j \frac{4\pi e_j^2}{m_j^2 c} \int dH\ dP_\theta\ \frac{2\pi}{\omega_a-\omega_b} \sum_m J_{\ell+m}(\lambda_{n\ell}b)\ J_m(\lambda_{n\ell}a)$$

$$\cdot\ [\frac{\partial f_{oj}}{\partial H}\ \{U\} + \frac{\partial f_{oj}}{\partial P_\theta}\ \{V\}]\ \frac{1}{\ell\omega_b - m(\omega_a-\omega_b)-\omega}\ , \qquad (2.29)$$

with {U} and {V} given by (2.26) and (2.27).

To get the corresponding result for (2.24), we must first deal with the term $\nabla \times (\underline{v}f_{1j})$. Using

$$\nabla \times (\underline{v}f_{1j}) = \frac{1}{r}\ \frac{\partial}{\partial r}\ (rv_\theta f_{1j}) - \frac{1}{r}\ \frac{\partial}{\partial \theta}\ (v_r f_{1j})$$

and integrating by parts over r we have

$$(-\lambda'^2_{n\ell} + \frac{\omega^2}{c^2})(-i)\ \frac{R^2}{2}\ [1 - (\frac{\ell}{\lambda_{n\ell}R})^2]\ J^2_\ell(\lambda'_{n\ell}R)\ \lambda'_{n\ell}\alpha_{n\ell}$$

$$= - \sum_j \frac{4\pi e_j}{c} \int drr\ J_\ell(\lambda'_{n\ell}r)\ e^{-i\ell\theta} \int d^2v\ [\frac{v_\theta}{r}\frac{\partial}{\partial r} - \frac{v_r}{r}(i\ell)]\ f_{1j}$$

$$= - \frac{i\lambda'_{n\ell}}{2} \sum_j \frac{4\pi e_j}{c} \int drr \int d^2v\ [v_+\ J_{\ell+1}(\lambda'_{n\ell}r)\ e^{-i(\ell+1)\theta}$$

$$+ v_-\ J_{\ell-1}(\lambda'_{n\ell}r)\ e^{-i(\ell-1)\theta}]\ f_{1j}\ . \tag{2.30}$$

Now we can make the change of variables of integration

$$rdrd^2v \to \frac{1}{m_j^2}\ dH_\perp dP_\theta dt$$

and obtain

$$(-\lambda'^2_{n\ell} + \frac{\omega^2}{c^2})\ \frac{R^2}{2}\ [1 - (\frac{\ell}{\lambda_{n\ell}R})^2]\ J^2_\ell(\lambda'_{n\ell}R)\ \alpha_{n\ell}$$

$$= \sum_j \frac{4\pi e_j}{m_j^2 c} \int dH_\perp dP_\theta dt \sum_m (-1)^m\ [i\omega_a a\ J_{\ell+m}(\lambda'_{n\ell}b)\ J'_m(\lambda'_{n\ell}a)$$

$$+ i\omega_b\ J'_{\ell+m}(\lambda_{n\ell}b)\ J_m(\lambda'_{n\ell}a)]\ e^{i[m(\omega_a-\omega_b)-\ell\omega_b]t}\ f_{1j}$$

$$= \sum_j \frac{4\pi e_j^2}{m_j^2 c} \int dH_\perp dP_\theta\ \frac{2\pi}{\omega_a-\omega_b} \sum_{m,n'} [\omega_a a\ J_{\ell+m}(\lambda'_{n\ell}b)\ J_m(\lambda'_{n\ell}a)$$

$$+ \omega_b b J'_{\ell+m}(\lambda'_{n\ell}b) J_m(\lambda'_{n\ell}a)]\cdot[\frac{\partial f_{oj}}{\partial H_\perp}\{U\} + \frac{\partial f_{oj}}{\partial P_\theta}\{V\}]\ \frac{1}{\ell\omega_b - m(\omega_a-\omega_b)-\omega}\ . \tag{2.31}$$

Equations (2.29) and (2.31) represent a linear relation among the expansion coefficients which corresponds to Eq. (1.20) in the electrostatic case. As in that case, it is useful to write the relation in terms of a dispersion matrix $D(\omega)$. If we write (2.31) as

$$\sum_{n'} D^{\alpha\alpha}_{nn'}(\omega)\ \alpha_{n'} + \sum_{n'} D^{\alpha\beta}_{nn'}(\omega)\ \beta_{n'} = 0$$

and (2.29) as

$$\sum_{n'} D^{\beta\alpha}_{nn'}(\omega)\ \alpha_{n'} + \sum_{n'} D^{\beta\beta}_{nn'}(\omega)\ \beta_{n'} = 0$$

we may combine both equations into a dispersion relation in matrix form:

$$\left(\begin{array}{c|c} D^{\alpha\alpha}_{nn'}(\omega) & D^{\alpha\beta}_{nn'}(\omega) \\ \hline D^{\beta\alpha}_{nn'}(\omega) & D^{\beta\beta}_{nn'}(\omega) \end{array}\right) \left(\begin{array}{c} \alpha_{n'} \\ \hline \beta_{n'} \end{array}\right) = 0 \quad . \tag{2.32}$$

The "normal modes" of the plasma are again found by truncating D for some suitable range of n's and solving the equation

$$\det[D(\omega)] = 0 \ .$$

For each root ω_0 the eigenvector of $D(\omega_0)$ with zero eigenvalue gives the expansion of the potentials for the corresponding mode according to (2.14).

For future reference, we write the submatrices of (2.32) explicitly as

$$D^{\alpha\alpha}_{nn'}(\omega) = \left(\frac{\omega^2}{c^2} - \lambda'^2_{n\ell}\right) \frac{R^2}{2} \left[1 - \left(\frac{\ell}{\lambda'_{n\ell}R}\right)^2\right] J^2_\ell(\lambda'_{n\ell}R)\, \delta_{nn'}$$

$$-\sum_j \frac{4\pi e_j^2}{m_j^2 c} \int dH_\perp dP_\theta \frac{2\pi}{\omega_a-\omega_b} \sum_m [\omega_a a J_{\ell+m}(\lambda'_{n\ell}b) J'_m(\lambda'_{n\ell}a) + \omega_b b J'_{\ell+m}(\lambda'_{n\ell}b) J_m(\lambda'_{n\ell}a)]$$

$$\cdot \left\{\frac{\partial f_{oj}}{\partial H_\perp} \left[\frac{\omega\omega_a a}{c} J_{\ell+m}(\lambda'_{n'\ell}b)\, J'_m(\lambda'_{n'\ell}a) + \frac{\omega\omega_b b}{c} J'_{\ell+m}(\lambda'_{n'\ell}b)\, J_m(\lambda'_{n'\ell}a)\right]\right.$$

$$+ \frac{\partial f_{oj}}{\partial P_\theta} \left[\frac{\omega}{c} \{aJ_{\ell+m}(\lambda'_{n'\ell}b)\, J'_m(\lambda'_{n'\ell}a) + bJ'_{\ell+m}(\lambda'_{n'\ell}b)\, J_m(\lambda'_{n'\ell}a)\}\right.$$

$$\left.\left. + \frac{\omega_a-\omega_b}{c} \{mbJ'_{\ell+m}(\lambda'_{n'\ell}b)\, J_m(\lambda'_{n'\ell}a) + (\ell+m)\, aJ_{\ell+m}(\lambda'_{n'\ell}b)\, J_m(\lambda'_{n'\ell}a)\}\right]\right\}$$

$$\cdot \frac{1}{\ell\omega_b - m(\omega_a-\omega_b) - \omega} \quad , \tag{2.33}$$

$$D^{\alpha\beta}_{nn'}(\omega) = \sum_j \frac{4\pi e_j^2}{m_j^2 c} \int dH_\perp dP_\theta \frac{2\pi}{\omega_a-\omega_b} \sum_m \frac{c}{\lambda_{n'\ell}\omega} \left(-\lambda^2_{n'\ell} + \frac{\omega^2}{c^2}\right)$$

$$\cdot [\omega_a a\, J_{\ell+m}(\lambda'_{n\ell}b)\, J'_m(\lambda'_{n\ell}a) + \omega_b b\, J'_{\ell+m}(\lambda'_{n\ell}b)\, J_m(\lambda'_{n\ell}a)]$$

$$\cdot \left\{\frac{\partial f_{oj}}{\partial H_\perp} [\ell\omega_b - m(\omega_a-\omega_b)]\, J_{\ell+m}(\lambda_{n'\ell}b)\, J_m(\lambda_{n'\ell}a)\right.$$

$$\left. + \ell \frac{\partial f_{oj}}{\partial P_\theta} J_{\ell+m}(\lambda_{n'\ell}b)\, J_m(\lambda_{n'\ell}a)\right\} \frac{1}{\ell\omega_b - m(\omega_a-\omega_b) - \omega} \quad , \tag{2.34}$$

$$D^{\beta\alpha}_{nn'}(\omega) = \omega \sum_j \frac{4\pi e_j^2}{m_j^2 c} \int dH_\perp dP_\theta \frac{2\pi}{\omega_a-\omega_b} \sum_m J_{\ell+m}(\lambda_{n\ell}b)\ J_m(\lambda_{n\ell}a)$$

$$\cdot \left\{\frac{\partial f_{oj}}{\partial H_\perp} [\frac{\omega\omega_a a}{c} J_{\ell+m}(\lambda'_{n'\ell}b)\ J'_m(\lambda'_{n'\ell}a) + \frac{\omega\omega_b b}{c} J'_{\ell+m}(\lambda'_{n'\ell}b)\ J_m(\lambda'_{n'\ell}a)]\right.$$

$$+ \frac{\partial f_{oj}}{\partial P_\theta} [\frac{\omega a}{c} J_{\ell+m}(\lambda'_{n'\ell}b)\ J'_m(\lambda'_{n'\ell}a) + \frac{\omega b}{c} J'_{\ell+m}(\lambda'_{n'\ell}b)\ J_m(\lambda'_{n'\ell}a)$$

$$\left. + \frac{\omega_a-\omega_b}{c} \{mb\ J'_{\ell+m}(\lambda'_{n'\ell}b)\ J_m(\lambda'_{n'\ell}a) + (\ell+m)\ aJ'_{\ell+m}(\lambda'_{n'\ell}b)\ J_m(\lambda'_{n'\ell}a)\}]\right\}$$

$$\cdot \frac{1}{\ell\omega_b-m(\omega_a-\omega_b)-\omega} \quad , \tag{2.35}$$

$$D^{\beta\beta}_{nn'}(\omega) = (\frac{\omega^2}{c^2} - \lambda^2_{n\ell})\ \lambda_{n\ell}\ \frac{R^2}{2}\ J^2_{\ell+1}(\lambda_{n\ell}R)\ \delta_{nn'}$$

$$\cdot \int dH_\perp dP_\theta \frac{2\pi}{\omega_a-\omega_b} \sum_m J_{\ell+m}(\lambda_{n\ell}b)\ J_m(\lambda_{n\ell}a) \left\{\frac{\partial f_{oj}}{\partial H_\perp} [\ell\omega_b-m(\omega_a-\omega_b)] + \ell\ \frac{\partial f_{oj}}{\partial P_\theta}\right\}$$

$$\cdot \frac{J_{\ell+m}(\lambda_{n'\ell}b)J_m(\lambda_{n'\ell}a)}{\ell\omega_b-m(\omega_a-\omega_b)-\omega} \quad . \tag{2.36}$$

As a check on the consistency of our results, we now wish to show that for the limit $c \to \infty$ in Eqs. (2.32) - (2.36), we recover the electrostatic dispersion relation of Chapter I. Using (1.19), (1.38) and (1.40), and taking $k_z = 0$, we may write the electrostatic dispersion relation as

$$\det[D^{ES}_{nn'}(\omega) = 0]$$

where the matrix D^{ES} is given by

$$D^{ES}_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{m_j^2 \lambda_{n\ell}^2} \frac{2}{R^2 J_{\ell+1}(\lambda_{n\ell}R) J_{\ell+1}(\lambda_{n'\ell}R)} \int dH\, dP_\theta \frac{2\pi}{\omega_a - \omega_b}$$

$$\cdot \sum_m J_{\ell+m}(\lambda_{n\ell}b)\, J_m(\lambda_{n\ell}a) \left\{ \frac{\partial f_{oj}}{\partial H_\perp} [\ell\omega_b - m(\omega_a - \omega_b)] + \ell \frac{\partial f_{oj}}{\partial P_\theta} \right\}$$

$$\cdot \frac{J_{\ell+m}(\lambda_{n'\ell}b) J_m(\lambda_{n'\ell}a)}{\ell\omega_b - m(\omega_a - \omega_b) - \omega} \quad . \tag{2.37}$$

As we let $c \to \infty$ in (2.33) - (2.36), we find that $D^{\alpha\alpha}$ becomes a constant diagonal matrix and $D^{\beta\alpha} \to 0$. Thus the dispersion relation becomes

$$\lim_{c\to\infty} \det[D_{nn'}(\omega)] = \det[\lim_{c\to\infty} D_{nn'}(\omega)] \cdot \det[\lim_{c\to\infty} D_{nn'}(\omega)] = 0 \ . \tag{2.38}$$

For $\mathrm{Im}(\omega) > 0$,

$$\det[\lim_{c\to\infty} D_{nn'}(\omega)] \neq 0 \ ,$$

so we may factor $D^{\alpha\alpha}_{nn'}$ out of (2.86) and the unstable modes are determined by $D^{\beta\beta}$:

$$\det[\lim_{c\to\infty} D_{nn'}(\omega)] = 0 \ . \tag{2.38}$$

If we define

$$D^{EM}_{nn'}(\omega) = - \frac{2}{\lambda_{n\ell}^3 R^2 J_{\ell+1}^2(\lambda_{n\ell}R)} \lim_{c\to\infty} D^{\beta\beta}_{nn'}(\omega) \tag{2.40}$$

we may alternatively write the dispersion relation as

$$\det[D^{EM}_{nn'}(\omega)] = 0 \ , \tag{2.41}$$

where from (2.36) and (2.40)

$$D^{EM}_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{m_j^2\lambda_{n\ell}^2} \left(\frac{\lambda_{n'\ell}}{\lambda_{n\ell}}\right) \frac{2}{R^2 J^2_{\ell+1}(\lambda_{n\ell}R)} \int dH_\perp dP_\theta \frac{2\pi}{\omega_a-\omega_b}$$

$$\cdot \sum_m J_{\ell+m}(\lambda_{n\ell}b)\, J_m(\lambda_{n\ell}a) \left\{\frac{\partial f_{oj}}{\partial H_\perp}[\ell\omega_b - m(\omega_a-\omega_b)] + \ell \frac{\partial f_{oj}}{\ell P_\theta}\right\}$$

$$\cdot \frac{J_{\ell+m}(\lambda_{n'\ell}b) J_m(\lambda_{n'\ell}a)}{\ell\omega_b - m(\omega_a-\omega_b)-\omega} \ . \tag{2.42}$$

Comparing (2.42) and (2.37) we see that

$$D^{EM}_{nn'}(\omega) = D^{ES}_{nn'}(\omega) \cdot \frac{\lambda_{n'\ell}J_{\ell+1}(\lambda_{n'\ell}R)}{\lambda_{n\ell}J_{\ell+1}(\lambda_{n\ell}R)} \ . \tag{2.43}$$

Now suppose we are given an arbitrary matrix $M = (M_{nn'})$ and an arbitrary set of nonzero numbers a_n, and consider the matrix $M' = (M_{nn'})$, where

$$M'_{nn'} = \frac{a_{n'}}{a_n} M_{nn'} \ .$$

Thus det{M'} consists of a sum of products of elements of M', each product containing exactly one element from every row and column of M'. Consequently, each a_k appears exactly once in the denominator and once in the numerator of this product, so that all the a_k's cancel out in the product and we have

$$\det[M'_{nn'}] = \det[M_{nn'}] \ .$$

Since by (2.43) $D^{EM}_{nn'}(\omega)$ and $D^{ES}_{nn'}(\omega)$ are related in the same way as M and M' above, we see that the dispersion relations (2.41) and (2.36) are equivalent. Thus we have shown that as $c \to \infty$, the electromagnetic dispersion relation obtained in this chapter reduces to the electrostatic dispersion relation of Chapter I as it should.

III. STABILITY ANALYSIS OF A RELATIVISTIC E-LAYER

In this chapter we apply the formalism of the preceding chapter to the calculation of unstable interactions between a relativistic E-layer and a warm background plasma. Interest in this interaction stems from plasma confinement experiments such as Astron[9] and various microwave generation devices.[10,11] First we will use a fluid model with uniform geometry to elucidate the nature of the instability, and then give a fully kinetic, non-local treatment based on the methods of Chapter II.

The mode of interest was first analyzed by Striffler and Kammash[12] to account for radiation observed near the upper hybrid frequency in Astron.[13] They showed that extraordinary electromagnetic modes[14] of the background plasma may be driven unstable by resonant interaction with perturbations in the relativistic E-layer. We begin by outlining their derivation of the dispersion relation for this mode in the fluid approximation.

The geometry of the problem and the coordinate system to be used are shown in Figure (3.1). The thickness of the E-layer, t_{EL}, is approximately 20 cm for Astron, and its average radius is 40 cm. We assume the modes we are dealing with have wavelengths smaller than these dimensions (this assumption will be quantified below) so that we can use the uniform beam and plasma approximation. We consider a uniform cold plasma of density n_p immersed in a constant uniform magnetic field $\underline{B}_o$. The plasma will be described by the fluid equations:

$$\frac{\partial n_j(\underline{x},t)}{\partial t} + \nabla\cdot[n_j(\underline{x},t)\ \underline{v}_j(\underline{x},t)] = 0\ ,$$

$$m_j \frac{d}{dt} \underline{v}_j(\underline{x},t) = q[\underline{E}(\underline{x},t) + \frac{1}{c}\underline{v}_j(\underline{x},t) \times \underline{B}(\underline{x},t)]\ , \qquad (3.1)$$

$$\underline{J}_j = n_j\ q_j\ \underline{v}_j\ ,\ \frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v}_j \cdot \nabla\ .$$

where $n_j(\underline{x},t)$, $\underline{v}_j(\underline{x},t)$, m_j, and q_j are the density, velocity, particle mass and charge, respectively, of species j. We are interested in high frequency modes so we ignore ion motion and deal only with the plasma electrons.

We consider the beam particles to have essentially straight-line orbits in the local approximation and ignore the effect of the external magnetic field on the beam. Thus the E-layer consists in this approximation of an infinite monoenergetic homogeneous beam of density n_B and relativistic velocity $\underline{V}_B$. However, we do assume that perturbations in the beam are periodic in the direction of beam motion, with periodicity length $2\pi R$. We neglect self-fields and collisions, so that the beam is described by the relativistic fluid equations:

$$\frac{\partial}{\partial t} n(\underline{x},t) + \nabla\cdot[n(\underline{x},t)\ \underline{v}(\underline{x},t)] = 0\ ,$$

$$\frac{d}{dt}[mv(\underline{x},t)] = q[\underline{E}(\underline{x},t) + \frac{1}{c}\underline{v}(\underline{x},t) \times \underline{B}_1(\underline{x},t)]\ , \qquad (3.2)$$

$$\underline{J} = nq\underline{v}\ ,\ \frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla\ .$$

Here $\underline{B}_1$ represents only the perturbed field, and

$$m = m_o(1 - \frac{\underline{v}\cdot\underline{v}}{c^2})^{-1/2} = m_o\gamma \quad ,$$

where m_o is the rest mass of the electron and γ is the relativistic mass factor.

The electromagnetic field is taken to be a plane wave:

$$\underline{E}(\underline{x},t) = \underline{\varepsilon}(\underline{k},\omega)\ e^{i\underline{k}\cdot\underline{x}-i\omega t} \quad ,$$

$$\underline{B}(\underline{x},t) = \underline{b}(\underline{k},\omega)\ e^{i\underline{k}\cdot\underline{x}-i\omega t} + \underline{B}_o \quad ,$$

and from Maxwell's equations we obtain the wave equation:

$$\underline{k}\times(\underline{k}\times\underline{\varepsilon}) + \frac{\omega^2}{c^2}\underline{\varepsilon} = -\frac{4\pi i\omega}{c^2}\underline{J}(\underline{\varepsilon}) \quad , \tag{3.3}$$

where $\underline{J}(\underline{\varepsilon})$ is the induced current density and can be written as the sum of contributions from the plasma and from the beam:

$$\underline{J}(\underline{\varepsilon}) = \underline{J}_{BEAM}(\underline{\varepsilon}) + \underline{J}_{PLASMA}(\underline{\varepsilon}) \quad .$$

We wish to perform a linearized stability analysis of the beam-plasma system, so we calculate the induced current densities from the linearized versions of (3.1) and (3.2). For the plasma, the result is

$$\frac{4\pi i\omega}{c^2} J_{1P} = -\frac{\omega_{PP}^2}{c^2}\ \frac{\omega^2\underline{\varepsilon} - i\omega\underline{\omega}_{CP}\times\underline{\varepsilon} - (\underline{\omega}_{CP}\cdot\underline{\omega})\underline{\omega}_{CP}}{\omega^2 - \omega_{CP}^2} \quad , \tag{3.4}$$

where

$$\omega_{PP} = \left(\frac{4\pi q_P^2 n_P}{m_o}\right)^{1/2} \quad , \quad \omega_{CP} = |\omega_{CP}| = \frac{q_P B_o}{m_o C} \quad ,$$

are the electron plasma and cyclotron frequencies, respectively.

For the beam we obtain

$$\frac{4\pi i\omega}{c^2} \underline{J}_{1B} = - \frac{\omega_{PB}^2}{c^2(\omega-\underline{k}\cdot\underline{V}_B)^2} \; (\omega-\underline{k}\cdot\underline{V}_B) \; [\omega\underline{\varepsilon} + \underline{V}_B \times (\underline{k}\times\underline{\varepsilon}) - \omega \frac{(\underline{V}_B\cdot\underline{\varepsilon})}{c^2} \underline{V}_B]$$

$$+ \underline{V}_B\underline{k}\cdot[\omega\underline{\varepsilon} + \underline{V}_B \times (\underline{k}\times\underline{\varepsilon}) - \omega \frac{(\underline{V}_B\cdot\underline{\varepsilon})}{c^2} \underline{V}_B]\} \; , \qquad (3.5)$$

where

$$\omega_{PB} = \left(\frac{4 \; q_B^2 n_B}{m_o \gamma_B}\right)^{1/2}$$

is the beam plasma frequency and

$$\gamma_B = \left(1 - \frac{V_B^2}{c^2}\right)^{-1/2} \quad .$$

Combining (3.4) and (3.5) the wave equation (3.3) becomes

$$\underline{k} \times (\underline{k}\times\underline{\varepsilon}) + \frac{\omega^2}{c^2}\underline{\varepsilon} = \frac{\omega_{PP}^2}{c^2} \; \frac{\omega^2\underline{\varepsilon} - i\omega\underline{\omega}_{CP} \times \underline{\varepsilon} - (\underline{\omega}_{CP}\cdot\underline{\varepsilon})\underline{\omega}_{CP}}{\omega^2 - \omega_{CP}^2}$$

$$+ \frac{\omega_{PB}^2}{c^2(\omega-\underline{k}\cdot\underline{V}_B)^2} \{(\omega-\underline{k}\cdot\underline{V}_B) \; [\omega\underline{\varepsilon} + \underline{V}_B \times (\underline{k}\times\underline{\varepsilon}) - \omega \frac{(\underline{V}_B\cdot\underline{\varepsilon})}{c^2} \underline{V}_B$$

$$+ \underline{V}_B\underline{k}\cdot[\omega\underline{\varepsilon} + \underline{V}_B \times (\underline{k}\times\underline{\varepsilon}) - \omega \frac{(\underline{V}_B\cdot\underline{\varepsilon})}{c^2} \underline{V}_B]\} \; . \qquad (3.6)$$

We wish to consider modes with $\underline{k}$ perpendicular to the magnetic field. Taking $\underline{B}_o$ in the z-direction, $\underline{V}_B$ in the y-direction we have

$$\underline{k} = (k_x, k_y, 0) \quad , \quad \underline{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_z) \quad , \quad \underline{V}_B = (0, V_B, 0) \; ,$$

and the x, y, and z components of (3.6) become

$$k_y(k_x\varepsilon_y - k_y\varepsilon_x) + \frac{\omega^2}{c^2}\varepsilon_x = \frac{\omega_{PP}^2}{c^2}\,\frac{\omega^2\varepsilon_x + i\omega\omega_{CP}\varepsilon_y}{\omega^2 - \omega_{CP}^2} + \frac{\omega_{PB}^2}{c^2}\,\frac{(\omega - k_yV_B)\varepsilon_x + k_xV_B\varepsilon_y}{\omega - k_yV_B} \quad ,$$

$$-k_x(k_x\varepsilon_y - k_y\varepsilon_x) + \frac{\omega^2}{c^2}\varepsilon_y = \frac{\omega_{PP}^2}{c^2}\,\frac{\omega^2\varepsilon_y - i\omega\omega_{CP}\varepsilon_x}{\omega^2 - \omega_{CP}^2}$$

$$+ \frac{\omega_{PB}^2}{c^2}\,\frac{k_xV_B(\omega - k_yV_B)\varepsilon_x + \left(\frac{\omega^2}{\gamma_B^2} + k_x^2V_B^2\right)\varepsilon_y}{(\omega - k_yV_B)^2} \quad ,$$

$$-k_\perp^2\varepsilon_z + \frac{\omega^2}{c^2}\varepsilon_z = \frac{\omega_{PP}^2}{c^2}\varepsilon_z + \frac{\omega_{PP}^2}{c^2}\varepsilon_z \; , \quad k_\perp^2 = k_x^2 + k_y^2 \; .$$

One solution is $\varepsilon_x = \varepsilon_y = 0$, $\varepsilon_z \neq 0$, which represents a purely electromagnetic model (called the ordinary mode[14]) with dispersion relation

$$\omega^2 = k_\perp^2c^2 + \omega_{PP}^2 + \omega_{PB}^2 \; .$$

Clearly this mode is stable. The other solution to the wave equation has ε_x, $\varepsilon_y \neq 0$; $\varepsilon_z = 0$. The corresponding dispersion relation is

$$[-k_y^2c^2 + \omega^2 - \frac{\omega^2\omega_{PP}^2}{\omega^2-\omega_{CP}^2} - \omega_{PB}^2]\cdot\left[-k_x^2c^2 + \omega^2 - \frac{\omega^2\omega_{PP}^2}{\omega^2-\omega_{CP}^2} - \omega_{PB}^2 \frac{\frac{\omega^2}{\gamma_B^2} + k_x^2V_B^2}{(\omega-k_yV_B)^2}\right]$$

$$-\left[k_xk_yc^2 + i\,\frac{\omega\omega_{CP}\omega_{PP}^2}{\omega^2-\omega_{CP}^2} - \frac{\omega_{PB}^2k_xV_B}{\omega-k_yV_B}\right]\cdot\left[k_xk_yc^2 - i\,\frac{\omega\omega_{CP}\omega_{PP}^2}{\omega^2-\omega_{CP}^2} - \frac{\omega_{PB}^2k_xV_B}{\omega-k_yV_B}\right] = 0\ . \tag{3.7}$$

If we set the beam density to zero ($\omega_{PB} = 0$), we get the dispersion relation for the extraordinary mode of a cold plasma:

$$(k_\perp c)^2 = \frac{\omega_k^2(\omega_k^2-\omega_{CP}^2-2\omega_{PP}^2) + \omega_{PP}^4}{\omega_k^2 - \omega_{CP}^2 - \omega_{PP}^2}\ .$$

Plots of this equation for $\omega_{CP} > \omega_{PP}$ and for $\omega_{CP} < \omega_{PP}$ are shown in Fig. (3.2). There are two branches, one above and one below the upper hybrid frequency $\omega_H = (\omega_{CP}^2+\omega_{PP}^2)^{1/2}$. The upper branch is also above the velocity of light line, so that no resonant interaction with particles can occur. We expect the unstable modes to occur near resonance with the beam, i.e.,

$$\omega \sim k_yV_B \sim \omega_k\ .$$

If we let ℓ be the azimuthal model number for the beam, we have

$$k_yV_B = \frac{2\pi}{\lambda_y} R\omega_{CB} \equiv \ell\omega_{CB}\ .$$

For the validity of the localized plane wave approximation we must assume the wavelength in the radial or x-direction is smaller than the smallest relevant radial dimension of Astron, which we take to be t_{EL}, the thickness of the E-layer. Thus we assume $\lambda_x << t_{EL}$ or for typical Astron parameters t_{EL} = 20 cm, $\omega_{CB} = 2 \times 10^8$ Hz,

$$\frac{k_x c}{\omega_{CB}} >> \left(\frac{k_x c}{\omega_{CB}}\right)_{min} = \frac{2\pi c}{t_{EL}\omega_{CB}} \cong 15 \ . \tag{3.8}$$

The condition on the total perpendicular wave vector k is then

$$\frac{k_\perp c}{\omega_{CB}} = \left(\frac{k_x^2 c^2}{\omega_{CB}^2} + \frac{k_y^2 c^2}{\omega_{CB}^2}\right)^{1/2} \cong \left(\frac{k_x^2 c^2}{\omega_{CB}^2} + \ell^2\right)^{1/2} \cong \frac{k_x c}{\omega_{CB}} >> 1 \tag{3.9}$$

for $k_x >> k_y$.

For the limit of large $k_x c$ the dispersion relation (3.7) becomes

$$(\omega^2-\omega_H^2)(\omega-k_y V_B)^2 - \frac{\omega_{PB}^2}{\gamma_B^2}(\omega^2-\omega_H^2+\gamma_B^2\omega_{PP}^2) = 0 \ . \tag{3.10}$$

The frequencies and growth rates for the resulting modes are shown in Fig. (3.3) for a range of plasma densities.

Striffler and Kammash also give a kinetic treatment of this problem: their results will be discussed in comparison with the results to be obtained later in this chapter.

Next we will apply the methods of Chapter II to treat this problem taking into account the effects of finite geometry. We treat the beam and plasma as separate species, and consider first the dispersion relation for the background plasma. Since we are taking $k_z = 0$, we ignore the z-dependence of the distribution function and take f_o to be a two-dimensional Maxwellian

$$f_o = \frac{m_p \hat{n}}{2\pi T} e^{-H_\perp/T} , \tag{3.11}$$

where $\hat{n}$ and T are the density and temperature of the plasma. Since there is no electric field in equilibrium, we have

$$H_\perp = \frac{1}{2} m_p v_\perp^2 .$$

The only force acting on the plasma particles in equilibrium is due to the magnetic field, so the particles simply move in circles in the x-y plane at frequency ω_c. Thus for the plasma electrons we have

$$v_\perp = -\omega_c a ,$$

where a is the electron gyroradius, and

$$f_o(v_\perp) = \frac{\hat{n} m_p}{2\pi T} e^{-\frac{mv_\perp^2}{2T}} . \tag{3.12}$$

We now write (2.33) - (2.36) in the form

$$D^{\alpha\alpha}_{nn'}(\omega) = \left(\frac{\omega^2}{c^2} - \lambda'^2_{n\ell}\right) \frac{R^2}{2} \left[1 - \left(\frac{\ell}{\lambda'_{n\ell} R}\right)^2\right] J^2_\ell(\lambda'_{n\ell} R)\, \delta_{nn'} + X^{\alpha\alpha}_{nn'}(\omega) ,$$

$$D^{\alpha\beta}_{nn'}(\omega) = X^{\alpha\beta}_{nn'}(\omega) \quad , \quad D^{\beta\alpha}_{nn'}(\omega) = X^{\beta\alpha}_{nn'}(\omega) \quad ,$$

$$D^{\beta\beta}_{nn'}(\omega) = (\frac{\omega^2}{c^2} - \lambda^2_{n\ell})\, \lambda_{n\ell}\, \frac{R^2}{2}\, J^2_{\ell+1}(\lambda_{n\ell}R)\, \delta_{nn'} + X^{\beta\beta}_{nn'}(\omega) \quad . \tag{3.13}$$

With $\omega_a = -\omega_c$, $\omega_b = 0$, we have

$$X^{\alpha\alpha}_{nn'}(\omega) = -\frac{4\pi e^2}{m_p^2 c} \int dH_\perp dP_\theta \frac{2\pi}{|\omega_c|} \sum_m [-\omega_c a J_{\ell+m}(\lambda'_{n\ell}b)\, J'_m(\lambda'_{n\ell}a)] \frac{\partial f_o}{\partial H_\perp}$$
$$\cdot [-\frac{\omega\omega_c a}{c} J_{\ell+m}(\lambda'_{n'\ell}b)\, J'_m(\lambda'_{n'\ell}a)] \frac{1}{m\omega_c - \omega} \quad . \tag{3.14}$$

Next we reintroduce the variables a, b from Chapter I. Recall that a is the gyroradius of a particle and b is the distance of the gyrocenter from the origin. We change the variables of integration from $dH_\perp dP_\theta$ to dadb using (1.44), which for the present case becomes

$$\left|\frac{\partial(H_\perp, P_\theta)}{\partial(a,b)}\right| = m_p^2 |\omega_c|^3\, ab \ .$$

Equation (3.14) thus becomes

$$X^{\alpha\alpha}_{nn'}(\omega) = -\frac{4\pi e^2}{m_p^2 c} m_p^2 |\omega_c|^3 \int_o^R daa \int_o^{R-a} dbb \frac{2\pi}{|\omega_c|} \sum_m \frac{\partial f_o}{\partial H_\perp} \frac{\omega\omega_c^2 a^2}{c}$$
$$\cdot J_{\ell+m}(\lambda'_{n\ell}b) J'_m(\lambda'_{n\ell}a) J_{\ell+m}(\lambda'_{n'\ell}b) J'_m(\lambda'_{n'\ell}a) \frac{1}{m\omega_c - \omega} \quad . \tag{3.15}$$

In terms of a, the equilibrium distribution function may be written

$$f_o(v_\perp) = \frac{\hat{n}m_p}{2\pi T}\, e^{-\frac{m_p\omega_c^2 a^2}{2T}} .$$

We assume that the temperature T of the background plasma is low enough that the plasma electrons are much less energetic than the beam electrons. Consequently, f_o is essentially zero unless $a \ll R$, and we may take the integral over b in (3.15) to be from 0 to R, and the integral over a from 0 to ∞. Defining

$$B^{\alpha\alpha}_{nn'm} = \int_o^R dbbJ_{\ell+m}(\lambda'_{n\ell}b)\; J_{\ell+m}(\lambda'_{n'\ell}b) \tag{3.16}$$

we have

$$\chi^{\alpha\alpha}_{nn'}(\omega) = -\frac{4\pi e^2}{c^2}\,\omega\omega_c^4 \int_o^\infty daa^3\left(-\frac{m_p\hat{n}}{T^2} e^{-\frac{m_p\omega_c^2 a^2}{2T}}\right)\sum_m J'_m(\lambda'_{n\ell}a)J'_m(\lambda'_{n'\ell}a)\frac{B^{\alpha\alpha}_{nn'm}}{m\omega_c-\omega} . \tag{3.17}$$

Using the Bessel function identity

$$\int_o^\infty dxx\; e^{-a^2x^2}\; J_n(px)\; J_n(qx) = \frac{1}{2a^2}\, e^{-\frac{p^2+q^2}{4a^2}}\; I_n\!\left(\frac{pq}{2a^2}\right) , \tag{3.18}$$

we can complete the integration in (3.17). Differentiating (3.18) with respect to p and q we have

$$\frac{\partial^2}{\partial p\partial q}\int_o^\infty dxx\; e^{-a^2x^2}\; J_n(px)\; J_n(qx) = \int_o^\infty dxx^3\; e^{-a^2x^2}\; J'_n(px)\; J'_n(qx)$$

$$= \frac{1}{8a^6}\, e^{-\frac{p^2+q^2}{4a^2}}\left[pqI_n\!\left(\frac{pq}{2a^2}\right)-(p^2+q^2)I'_n\!\left(\frac{pq}{2a^2}\right)+pqI''_n\!\left(\frac{pq}{2a^2}\right)\right]+\frac{I'_n\!\left(\frac{pq}{2a^2}\right)e^{-\frac{p^2+q^2}{4a^2}}}{4a^4} .$$

Applying this result to (3.17) we obtain after some algebra

$$\chi^{\alpha\alpha}_{nn'}(\omega) = \frac{4\pi e^2\hat{n}}{m_p c^2} \sum_m \frac{\omega}{m\omega_c - \omega}$$

$$\left\{ -2\left(\frac{T}{m_p\omega_c^2}\right)\left[\frac{\lambda'^2_{n\ell}+\lambda'^2_{n'\ell}}{2} I'_m\left(\frac{\lambda'_{n\ell}\lambda'_{n'\ell}T}{m_p\omega_c^2}\right) - \lambda'_{n\ell}\lambda'_{n'\ell} I_m\left(\frac{\lambda'_{n\ell}\lambda'_{n'\ell}T}{m_p\omega_c^2}\right)\right]\right.$$

$$\left. + m^2\left(\frac{m_p\omega_c^2}{\lambda'_{n\ell}\lambda'_{n'\ell}T}\right) I_m\left(\frac{\lambda'_{n\ell}\lambda'_{n'\ell}T}{m_p\omega_c^2}\right)\right\} B^{\alpha\alpha}_{nn'm}\, e^{-\frac{(\lambda'^2_{n\ell}+\lambda'^2_{n'\ell})T}{2m_p\omega_c^2}} . \tag{3.19}$$

Similarly, defining

$$B^{\alpha\beta}_{nn'm} = \int_0^R dbb\; J_{\ell+m}(\lambda'_{n\ell}b)\; J_{\ell+m}(\lambda_{n'\ell}b) ,$$

$$B^{\beta\alpha}_{nn'm} = \int_0^R dbb\; J_{\ell+m}(\lambda_{n\ell}b)\; J_{\ell+m}(\lambda'_{n'\ell}b) ,$$

$$B^{\beta\alpha}_{nn'm} = \int_0^R dbb\; J_{\ell+m}(\lambda_{n\ell}b)\; J_{\ell+m}(\lambda_{n'\ell}b) ,$$

we have the results

$$\chi^{\alpha\beta}_{nn'}(\omega) = \left(\frac{\omega^2}{2} - \lambda^2_{n'\ell}\right) \frac{4\pi e^2\hat{n}}{\lambda_{n'\ell}\omega m_p} \sum_m \frac{m}{m\omega_c - \omega}$$

$$\cdot \left[\lambda_{n'\ell} I'_m\left(\frac{\lambda'_{n\ell}\lambda_{n'\ell}T}{m_p\omega_c^2}\right) - \lambda'_{n\ell} I_m\left(\frac{\lambda'_{n\ell}\lambda_{n'\ell}T}{m_p\omega_c^2}\right)\right] B^{\alpha\beta}_{nn'm}\, e^{-\frac{(\lambda'^2_{n\ell}+\lambda^2_{n'\ell})T}{2m_p\omega_c^2}} ,$$

$$\chi^{\beta\alpha}_{nn'}(\omega) = \frac{4\pi e^2\hat{n}}{m_p c^2} \frac{\omega^2}{\omega_c} \sum_m \frac{1}{m\omega_c-\omega}$$

$$\cdot \left[\lambda_{n\ell} I'_m\left(\frac{\lambda_{n\ell}\lambda'_{n'\ell}T}{m_p\omega_c^2}\right) - \lambda'_{n'\ell} I_m\left(\frac{\lambda_{n\ell}\lambda'_{n'\ell}T}{m_p\omega_c^2}\right)\right] B^{\beta\alpha}_{nn'm} e^{-\frac{(\lambda^2_{n\ell}+\lambda'^2_{n'\ell})T}{2m_p\omega_c^2}} ,$$

$$\chi^{\beta\beta}_{nn'}(\omega) = \left(\frac{\omega^2}{c^2} - \lambda^2_{n'\ell}\right) \frac{4\pi e^2\hat{n}}{\lambda_{n'\ell} m_p} \sum_m \frac{m\omega_c}{m\omega_c-\omega} \frac{m_p}{T} I_m\left(\frac{\lambda_{n\ell}\lambda_{n'\ell}T}{m_p\omega_c^2}\right) B^{\beta\beta}_{nn'm} e^{-\frac{(\lambda^2_{n\ell}+\lambda^2_{n'\ell})T}{2m_p\omega_c^2}} , \tag{3.20}$$

where the B's can be evaluated as follows:

$$B^{\alpha\alpha}_{nn'm} = \frac{R}{\lambda'^2_{n\ell}-\lambda'^2_{n'\ell}} [\lambda'_{n\ell}\, J_{\ell+m+1}(\lambda'_{n\ell}R)\, J_{\ell+m}(\lambda'_{n'\ell}R)$$

$$- \lambda'_{n'\ell}\, J_{\ell+m}(\lambda'_{n\ell}R)\, J_{\ell+m+1}(\lambda'_{n'\ell}R)] \quad n \neq n ,$$

$$B^{\alpha\beta}_{nn'm} = \frac{R}{\lambda'^2_{n\ell}-\lambda^2_{n'\ell}} [\lambda'_{n\ell}\, J_{\ell+m+1}(\lambda'_{n\ell}R)\, J_{\ell+m}(\lambda_{n'\ell}R)$$

$$- \lambda'_{n\ell} J_{\ell+m}(\lambda'_{n\ell}R)\, J_{\ell+m}(\lambda_{n'\ell}R)] , \tag{3.21}$$

$$B^{\alpha\beta}_{nn'm} = \frac{R}{\lambda^2_{n\ell}-\lambda'^2_{n'\ell}} [\lambda_{n\ell}\, J_{\ell+m+1}(\lambda_{n\ell}R)\, J_{\ell+m}(\lambda'_{n'\ell}R)$$

$$- \lambda'_{n'\ell}\, J_{\ell+m}(\lambda_{n\ell}R)\, J_{\ell+m+1}(\lambda'_{n'\ell}R)] ,$$

$$B^{\beta\beta}_{nn'm} = \frac{R}{\lambda^2_{n\ell}-\lambda^2_{n'\ell}} [\lambda_{n\ell}\, J_{\ell+m+1}(\lambda_{n\ell}R)\, J_{\ell+m}(\lambda_{n'\ell}R)$$

$$- \lambda_{n'\ell}\, J_{\ell+m}(\lambda_{n\ell}R)\, J_{\ell+m+1}(\lambda_{n'\ell}R)] \quad n \neq n' .$$

For the diagonal terms, where n = n', the above expressions for $B^{\alpha\alpha}$ and $B^{\beta\beta}$ should be replaced by

$$
\begin{aligned}
B^{\alpha\alpha}_{nnm} &= \frac{R^2}{2}\left[J^2_{\ell+m}(\lambda'_{n\ell}R) - J_{\ell+m-1}(\lambda'_{n\ell}R)\, J_{\ell+m+1}(\lambda'_{n\ell}R)\right] , \\
B^{\beta\beta}_{nnm} &= \frac{R^2}{2}\left[J^2_{\ell+m}(\lambda_{n\ell}R) - J_{\ell+m-1}(\lambda_{n\ell}R)\, J_{\ell+m+1}(\lambda_{n\ell}R)\right] .
\end{aligned}
\tag{3.22}
$$

To obtain the contribution of the E-layer to the dispersion matrix we must use a relativistic version of the analysis of Chapter II. First we determine a suitable distribution function. In the cold beam approximation, the E-layer consists of particles moving in circles with center at the origin and radii between r_{min} and r_{max}, where $r_{max} - r_{min} = t_{EL}$. To simplify the analysis, we will calculate the contribution to the susceptibility only for particles at one radius r_b; the total contribution of the beam will then be obtained by integrating over r_b from r_{min} to r_{max}. Since $k_z = 0$, we ignore the z-direction and P_z and write the equilibrium distribution function as a function of the remaining two constants of the motion: the energy and the azimuthal canonical momentum. For a particle with gyroradius r_b the energy is

$$
E = \gamma m_b c^2 \quad , \quad \gamma = \sqrt{1 + \left(\frac{\omega_c r_b}{c}\right)^2} . \tag{3.23}
$$

Here as usual ω_c is the non-relativistic signed cyclotron frequency for the particle. For a particle with gyro-orbit centered at the origin,

$$V_\theta = -\frac{r_b\omega_c}{\gamma} \quad ,$$

so that the angular momentum is given by

$$L = \gamma m_b r v_\theta + \frac{1}{2} r^2 m_b \omega_c = -\frac{1}{2} m_b \omega_c r_b^2 \ . \tag{3.24}$$

Thus the distribution function for these particles can be written

$$f_o(r,P_r,P_\theta) = A\delta[H(r,P_r,P_\theta)-E]\ \delta(P_\theta - L) \ . \tag{3.25}$$

To find the factor A we define the density of particles at radius r by

$$n(r) = \int dP_r dP_\theta f_o(r,P_r,P_\theta) \tag{3.26}$$

and require that

$$n(r) = \hat{n} r_b \delta(r-r_b) \tag{3.27}$$

so that

$$\int dr d\theta n(r) = \int dr d\theta dP_r dP_\theta f_o = 2\pi\hat{n} r_b \ . \tag{3.28}$$

Equation (3.28) represents the total number of particles in f_o, and $\hat{n}$ is a constant representing the particle density of the E-layer. To find A from (3.25) - (3.27), we need to evaluate the integral in (3.26). This is more easily accomplished in velocity space, so we convert the variables of integration from P_r and P_θ to v_r and v_θ.

Using

$$P_r = \gamma m v_r \quad , \quad P_\theta = \gamma m_b r_b v_\theta + \frac{1}{2} r_b^2 m_b \omega_c \quad , \quad v_r = 0 \ ,$$

we have for the Jacobian of the transformation:

$$\frac{\partial(P_r, P_\theta)}{\partial(v_r, v_\theta)} = \begin{vmatrix} \gamma m_b + m_b v_r \dfrac{\partial\gamma}{\partial v_r} & m_b v_r \dfrac{\partial\gamma}{\partial v_\theta} \\ \gamma^3 m_b r_b \dfrac{v_r v_\theta}{c^2} & \gamma^3 m_b r_b \dfrac{v_\theta^2}{c^2} + \gamma m_b r_b \end{vmatrix}$$

$$= \gamma^4 m_b^2 r_b \frac{v_\theta^2}{c^2} + \gamma^2 m_b^2 r_b = \gamma^4 m_b^2 r_b \ . \tag{3.29}$$

We again introduce the particle coordinates a, b of Chapter I, i.e., a is the gyroradius and b is the distance of the gyrocenter from the origin. When $r = a + b$, $v_\theta = - \dfrac{\omega_c a}{\gamma}$, so we have

$$\begin{gathered} H = \gamma m_b c^2 \quad , \quad \gamma = \sqrt{1 + \left(\frac{\omega_c a}{c}\right)^2} \ , \\ P_\theta = \gamma m_b r v_\theta + \frac{1}{2} r^2 m_b \omega_c = \frac{1}{2} m_b \omega_c (b^2 - a^2) \ . \end{gathered} \tag{3.30}$$

If we write

$$v_x = v \cos\phi \quad , \quad v_y = v \sin\phi \ ,$$

we have $v_\theta = v \sin(\phi - \theta)$, so

$$P_\theta = \gamma m r v \sin(\phi - \theta) + \frac{1}{2} m r^2 \omega_c \ , \tag{3.31}$$

which combined with (3.30) yields

$$\sin(\phi-\theta) = \frac{1}{\gamma mrv} \cdot \frac{1}{2} m\omega_c[b^2-a^2-r^2]$$
$$= -\frac{b^2-a^2-r^2}{2ar} \quad , \tag{3.32}$$

since $v = \frac{|\omega_c|a}{\gamma}$. Since $\frac{\partial H}{\partial v} = \gamma^3 mv$, we may write

$$\int d^2v\ \delta(H-E)\ \delta(P_\theta-L) = \int dvv \int d\phi\ \frac{1}{\gamma^3 mv}\ \delta(v-v(E))\ \delta(P_\theta-L)$$
$$= \frac{1}{m\gamma^3} \int d\phi\ \delta[\gamma mrv\ \sin(\phi-\theta) + \frac{1}{2} mr^2\omega_c-L] \ . \tag{3.33}$$

Now for all a, r we have $b \geq (r-a)$ so that $b^2 \geq r^2 - 2ar + a^2$ or $[a^2 + r^2 - b^2]/2ar \leq 1$. Thus (3.32) will be satisfied for two values of ϕ, which we denote by $\phi_1(L)$ and $\phi_2(L)$. We let $\phi(L)$ denote either $\phi_1(L)$ or $\phi_2(L)$ and write

$$|\cos\ [\phi_1(L)-\theta]| = |\cos[\phi_2(L)-\theta]| = |\cos[\phi(L)-\theta]|$$

so that (3.33) becomes

$$\int d^2v\delta(H-E)\delta(P_\theta-L) = \frac{1}{m\gamma^3}\ \frac{1}{|\gamma mrv\cos[\phi(L)-\theta]|} \int d\phi \Big[\delta[\phi-\phi_1(L)]+\delta[\phi-\phi_2(L)]\Big]$$
$$= \frac{1}{m\gamma^3}\ \frac{2}{|\gamma mrv\cos[\phi(L)-\theta]|} \quad ;$$

from (3.32) we have

$$|\cos[\phi(L)-\theta] = \sqrt{1 - \frac{b^2-a^2-r^2}{4a^2r^2}} = \frac{1}{2ar} \sqrt{[(r+a)^2-b^2][b^2-(r-a)^2]}$$

and

$$\int d^2v \; \delta(H-E) \; \delta(P_\theta - L) = \left(\frac{1}{m_b \gamma^3}\right)\left(\frac{1}{\gamma m_b r v}\right) \frac{4ar}{\sqrt{[(r+a)^2-b^2][b^2-(r-a)^2]}} \; . \tag{3.34}$$

Since we are dealing with a cold beam of radius r_b centered at the origin, we wish to take $a = r_b$ and $b = 0$ in (3.34). The right side of (3.34) is clearly singular for $b = 0$, as we expect from (3.27). To elucidate the nature of this singularity, we consider the limit as $b \to 0$:

$$\lim_{b \to o} \int d^2v \; \delta(H-E) \; \delta(P_\theta - L) = \frac{1}{\gamma^4 r v m_b^2} \lim_{b \to o} \frac{4ar}{\sqrt{(r+a)^2-b^2} \sqrt{b^2-(r-a)^2}}$$

$$= \frac{1}{\gamma^4 m_b^2 r v} \lim_{b \to o} \frac{2r}{\sqrt{b^2-(r-a)^2}} \; . \tag{3.35}$$

The right side of (3.35) will be real and non-zero only for $r = a$, at which point it will be infinite. It thus is proportional to a delta function in accordance with (3.27). To find the proportionality constant we write

$$\lim_{b \to o} \frac{r}{\sqrt{b^2-(r-a)^2}} = C \; \delta(r-a)$$

and find C by integrating over r from $a - b$ to $a + b$ (the limits for which the square root is real). Defining $r - a = s$, we have

$$\int_{a-b}^{a+b} \frac{rdr}{\sqrt{b^2-(r-a)^2}} = \int_{-b}^{b} \frac{(a+s)ds}{\sqrt{b^2-s^2}} = a \int_{-b}^{b} \frac{ds}{\sqrt{b^2-s^2}} = a\pi \ .$$

Thus $C = a\pi$ and

$$\int d^2v \ \delta(H-E) \ \delta(P_\theta-L) = \frac{2a\pi}{m_b^2\gamma^4 rv} \ \delta(r-a) = \frac{2\pi}{m_b^2\gamma^4 v} \ \delta(r-a) \ .$$

Using (3.29), (3.25), and (3.27), we then may write

$$\int dP_r dP_\theta f_o = A\gamma^4 m_b^2 r_b \int d^2v \ \delta(H-E) \ \delta(P_\theta-L) = \frac{2\pi Aa}{v} \ \delta(r-a) = \hat{n} r_b \ \delta(r-a).$$

Solving for A, we obtain

$$f_o(H,P_\theta) = \hat{n} \ \frac{v}{2\pi} \ \delta(H-E) \ \delta(P_\theta-L) = \hat{n} \ \frac{|\omega_c|a}{2\pi\gamma} \ \delta(H-E) \ \delta(P_\theta-L) \ . \tag{3.36}$$

Using (3.30), we may write the equilibrium distribution function in terms of a and b. The result is

$$f_o(a,b) = \frac{\hat{n}}{\pi\omega_c^2 m_b^2 b} \ \delta(a-r_b) \ \delta(b) \ . \tag{3.37}$$

If we write the Vlasov-Maxwell equations (2.1) and (2.2) in terms of the phase space variables $\underline{r}$, $\underline{P}$ rather than $\underline{r}$, $\underline{v}$, they remain valid relativistically. Thus to make (2.33) - (2.36) relativistically valid, we need only convert f_o from representing the number of particles in d^2rd^2v to the number of particles in d^2rdP_rdP, or

$$f_{oj} \rightarrow m_j^2 \ f_{oj} \ .$$

The susceptibilities due to the beam are then

$$y^{\alpha\alpha}_{nn'}(\omega) = -\frac{4\pi e^2}{c}\int dH_\perp dP_\theta \frac{2\pi}{|\omega_a-\omega_b|}\sum_m$$

$$\cdot\,[\omega_a a\, J_{\ell+m}(\lambda'_{n\ell}b)\, J'_m(\lambda'_{n\ell}a) + \omega_b b\, J_{\ell+m}(\lambda'_{n\ell}b)\, J_m(\lambda'_{n\ell}a)]$$

$$\cdot\left\{\frac{\partial f_o}{\partial H_\perp}\,[\frac{\omega\omega_a a}{c}\, J_{\ell+m}(\lambda'_{n'\ell}b)\, J'_m(\lambda'_{n'\ell}a) + \frac{\omega\omega_b b}{c}\, J'_{\ell+m}(\lambda'_{n'\ell}b)\, J_m(\lambda'_{n'\ell}a)]\right.$$

$$+\frac{\partial f_o}{\partial P_\theta}\,[\frac{\omega}{c}\,\{a\, J_{\ell+m}(\lambda'_{n'\ell}b)\, J'_m(\lambda'_{n'\ell}a) + b\, J'_{\ell+m}(\lambda'_{n'\ell}b)\, J_m(\lambda'_{n'\ell}a)$$

$$\left.+\frac{\omega_a-\omega_b}{c}\,\{mbJ'_{\ell+m}(\lambda'_{n'\ell}b)J_m(\lambda'_{n'\ell}a) + (\ell+m)aJ_{\ell+m}(\lambda'_{n'\ell}b)J_m(\lambda'_{n'\ell}a)\}]\right\}$$

$$\cdot\,\frac{1}{\ell\omega_b-m(\omega_a-\omega_b)-\omega}\ , \tag{3.38}$$

$$y^{\alpha\beta}_{nn'}(\omega) = \frac{4\pi e^2}{c}\int dH_\perp dP_\theta \frac{2\pi}{|\omega_a-\omega_b|}\sum_m \frac{c}{\lambda_{n'\ell}\omega}\,(-\lambda^2_{n'\ell} + \frac{\omega^2}{c^2})$$

$$\cdot\,[\omega_a a\, J_{\ell+m}(\lambda'_{n\ell}b)\, J'_m(\lambda'_{n\ell}a) + \omega_b b\, J'_{\ell+m}(\lambda'_{n\ell}b)\, J_m(\lambda'_{n\ell}a)]$$

$$\cdot\,\{\frac{\partial f_o}{\partial H_\perp}\,[\ell\omega_b-m(\omega_a-\omega_b)]\, J_{\ell+m}(\lambda_{n'\ell}b)\, J_m(\lambda_{n'\ell}a)$$

$$+\ \ell\,\frac{\partial f_o}{\partial P_\theta}\, J_{\ell+m}(\lambda_{n'\ell}b)\, J_m(\lambda_{n'\ell}a)\}\cdot\frac{1}{\ell\omega_b-m(\omega_a-\omega_b)-\omega}\ , \tag{3.39}$$

$$y^{\beta\alpha}_{nn'}(\omega) = \omega \frac{4\pi e^2}{c^2} \int dH_\perp dP_\theta \frac{2\pi}{|\omega_a - \omega_b|} \sum_m J_{\ell+m}(\lambda_{n\ell} b)\ J_m(\lambda_{n\ell} a)$$

$$\cdot \left\{ \frac{\partial f_o}{\partial H_\perp} [\frac{\omega\omega_a a}{c} J_{\ell+m}(\lambda'_{n'\ell} b)\ J'_m(\lambda'_{n'\ell} a) + \frac{\omega\omega_b b}{c} J'_{\ell+m}(\lambda'_{n'\ell} b)\ J_m(\lambda'_{n'\ell} a)] \right.$$

$$+ \frac{\partial f_o}{\partial P_\theta} [\frac{\omega a}{c} J_{\ell+m}(\lambda'_{n'\ell} b)\ J'_m(\lambda'_{n'\ell} a) + \frac{\omega b}{c} J'_{\ell+m}(\lambda'_{n'\ell} b)\ J_m(\lambda'_{n'\ell} a)$$

$$\left. + \frac{\omega_a - \omega_b}{c} \{mbJ'_{\ell+m}(\lambda'_{n'\ell} b)\ J_m(\lambda'_{n'\ell} a) + (\ell+m)\ aJ'_{\ell+m}(\lambda'_{n'\ell} b)\ J_m(\lambda'_{n'\ell} a)\}] \right\}$$

$$\cdot \frac{1}{\ell\omega_b - m(\omega_a - \omega_b) - \omega} \quad , \tag{3.40}$$

$$y^{\beta\beta}_{nn'}(\omega) = -\frac{4\pi e^2}{\lambda_{n'\ell}} (\frac{\omega^2}{c^2} - \lambda^2_{n'\ell}) \int dH_\perp dP_\theta \frac{2\pi}{|\omega_a - \omega_b|} \sum_m J_{\ell+m}(\lambda_{n\ell} b)\ J_m(\lambda_{n\ell} a)$$

$$\cdot \left\{ \frac{\partial f_o}{\partial H_\perp} [\ell\omega_v - m(\omega_a - \omega_b)] + \ell \frac{\partial f_o}{\partial P_\theta} \right\} \frac{J_{\ell+m}(\lambda_{n'\ell} b) J_m(\lambda_{n'\ell} a)}{\ell\omega_b - m(\omega_a - \omega_b) - \omega} \quad , \tag{3.41}$$

where y is used instead of χ to distinguish the E-layer susceptibility from that of the background plasma given by (3.19) - (3.20).

The next step is to substitute the beam distribution function (3.37) into (3.38) - (3.41) along with

$$\omega_a = -\frac{\omega_c}{\gamma} \quad , \quad \omega_b = 0 \ .$$

Since the calculations are very similar for all four matrix elements, we treat only the $\alpha\alpha$ element (3.38) in detail. First we express the constants of the motion in terms of a and b. We have

$$P_\theta = \frac{1}{2} m_b\omega_c(b^2-a^2) \quad , \quad \frac{\partial P_\theta}{\partial a} = -m_b\omega_c a \quad , \quad \frac{\partial P_\theta}{\partial b} = m_b\omega_c b \ . \qquad (3.42)$$

Thus

$$\frac{\partial(H_\perp,P_\theta)}{\partial(a,b)} = \frac{m_b^2|\omega_c|^3 ab}{\gamma} \quad , \qquad (3.43)$$

$$\frac{\partial f_o}{\partial H_\perp} = \frac{\gamma}{m\omega_c^2 a}\frac{\partial f_o}{\partial a} + \frac{\gamma}{m\omega_c^2 b}\frac{\partial f_o}{\partial b} \quad , \quad \frac{\partial f_o}{\partial P_\theta} = \frac{1}{m\omega_c b}\frac{\partial f_o}{\partial b} \quad , \qquad (3.44)$$

$$y^{\alpha\alpha}_{nn'}(\omega) = -\frac{4\pi e^2 m_b^2}{c}\int_o^R da \int_o^{R-a} dbab(2\pi\omega_c^2) \sum_m \frac{\omega_c a}{\gamma} J_{\ell+m}(\lambda'_{n\ell}b)\ J'_m(\lambda'_{n\ell}a)$$

$$\cdot \left\{\frac{\partial f_o}{\partial a}\frac{\omega}{m_b\omega_c c} J_{\ell+m}(\lambda'_{n'\ell}b)\ J'_m(\lambda'_{n'\ell}a) + \frac{\partial f_o}{\partial b}\right.$$

$$\cdot \left[\frac{(\ell+m)a}{\gamma m_b cb} J_{\ell+m}(\lambda'_{n'\ell}b)\ J'_m(\lambda'_{n'\ell}a) + \frac{m}{\gamma m_b c} J'_{\ell+m}(\lambda'_{n'\ell}b)\ J_m(\lambda'_{n'\ell}a)\right.$$

$$\left.\left. - \frac{\omega}{m_b\omega_c c} J'_{\ell+m}(\lambda'_{n'\ell}b)\ J_m(\lambda'_{n'\ell}a)\right]\right\} \frac{1}{m\frac{\omega_c}{\gamma}-\omega} \ . \qquad (3.45)$$

We write this as

$$y^{\alpha\alpha}_{nn'}(\omega,r_b) = \{X\} + \{Y\} \ ,$$

where {X} and {Y} are defined below, and the r_b argument indicates that this is the contribution from the E-layer particles of gyroradius r_b. Using the expression (3.37) for the equilibrium distribution function we have

$$\{X\} \equiv -\frac{4\pi e^2 m_b^2}{c}\int_0^R da \int_0^R dbab(2\pi\omega_c^2)\sum_m \frac{\omega_c a}{\gamma} J_{\ell+m}(\lambda'_{n\ell}b)\ J'_m(\lambda'_{n\ell}a)$$

$$\cdot \frac{\partial f_o}{\partial a}\frac{\omega}{m_b\omega_c c} J_{\ell+m}(\lambda'_{n'\ell}b)\ J'_m(\lambda'_{n'\ell}a)\frac{1}{m\frac{\omega_c}{\gamma}-\omega}$$

$$= -\frac{4\pi e^2\hat{n}}{c}\int_0^R da\int_0^R dba\sum_m \frac{\omega_c a}{\gamma} J_{\ell+m}(\lambda'_{n\ell}b)\ J'_m(\lambda'_{n\ell}a)$$

$$\cdot\ \delta'(a-r_b)\ \delta(b)\ J_{\ell+m}(\lambda'_{n'\ell}b)\ J'_m(\lambda'_{n'\ell}a)\frac{2\omega}{m_b\omega_c c}\frac{1}{m\frac{\omega_c}{\gamma}-\omega}\ .$$

Integrating by parts on a we have

$$\{X\} = \frac{4\pi e_b^2\hat{n}_b}{c}\int dadb\ \delta(b)\ \delta(a-r_b)\sum_m J_{\ell+m}(\lambda'_{n\ell}b)\ J_{\ell+m}(\lambda'_{n'\ell}b)$$

$$\cdot\frac{\partial}{\partial a}\left[\frac{\omega_c a^2}{\gamma} J'_m(\lambda'_{n\ell}a)\ J'_m(\lambda'_{n'\ell}a)\frac{1}{m\frac{\omega_c}{\gamma}-\omega}\right]\frac{2\omega}{m_b\omega_c c}\ .$$

Since $J_p(0)$ 0 only for $p = 0$, the integrals over a and b yield (remembering that $\int_0^R db\ \delta(b) = \frac{1}{2}$):

$$\{X\} = -\frac{4\pi e^2\hat{n}_b}{\gamma m_b c^2}\frac{\omega}{\ell\frac{\omega_c}{\gamma}+\omega}\frac{\partial}{\partial r_b}[r_b^2\ J'_\ell(\lambda'_{n\ell}r_b)\ J'_\ell(\lambda'_{n'\ell}r_b)]$$

$$-\frac{4\pi e^2\hat{n}_b}{m_b c^2}\omega r_b^2\ J'_\ell(\lambda'_{n\ell}r_b)\ J'_\ell(\lambda'_{n'\ell}r_b)\frac{\partial}{\partial r_b}\left[\frac{1}{\ell\omega_c+\omega\gamma}\right]\ .$$

Since

$$\gamma(r_b) = \sqrt{1+\frac{\omega_c^2 r_b^2}{c^2}}\ ,$$

we have

$$\frac{\partial}{\partial r_b}\left[\frac{1}{\ell\omega_c+\omega\gamma}\right] = -\frac{\omega\omega_c^2 r_b}{\gamma^3 c^2}\,\frac{1}{(\ell\,\frac{\omega_c}{\gamma}+\omega)^2}$$

and

$$\{X\} = -\frac{4\pi e^2\hat{n}_b}{\gamma m_b c^2}\,\frac{\omega}{\ell\,\frac{\omega_c}{\gamma}+\omega}\,\frac{\partial}{\partial r_b}\left[r_b^2\, J_\ell'(\lambda_{n\ell}' r_b)\, J'(\lambda_{n'\ell}' r_b)\right]$$

$$+\frac{4\pi e_b^2\hat{n}_b}{\gamma m_b c^2}\,\frac{\omega^2}{c^2}\, r_b\, J_\ell'(\lambda_{n\ell}' r_b)\, J_\ell'(\lambda_{n'\ell}' r_b)\,\frac{\omega_c^2 r_b^2}{\gamma^2}\,\frac{1}{(\ell\,\frac{\omega_c}{\gamma}+\omega)^2}\ . \tag{3.46}$$

The remainder of the right side of (3.45) is

$$\{Y\} = -\frac{4\pi e^2\hat{n}_b}{m_b c^2}\int dbb\,\frac{2\omega_c r_b}{\gamma}\sum_m J_{\ell+m}(\lambda_{n\ell}' b)\, J_m'(\lambda_{n\ell}' r_b)$$

$$\cdot\,\frac{\partial}{\partial b}\left[\frac{1}{b}\,\delta(b)\right]\cdot\left[-\frac{\omega}{\omega_c}\, J_{\ell+m}'(\lambda_{n'\ell}' b)\, J_m(\lambda_{n'\ell}' r_b)\right.$$

$$\left.+\frac{(\ell+m) r_b}{\gamma b}\, J_{\ell+m}(\lambda_{n'\ell}' b)\, J_m(\lambda_{n'\ell}' r_b) + \frac{m}{\gamma}\, J_{\ell+m}'(\lambda_{n'\ell}' b)\, J_m(\lambda_{n'\ell}' r_b)\right]$$

$$\cdot\,\frac{1}{m\,\frac{\omega_c}{\gamma}-\omega}\ . \tag{3.47}$$

Integrating by parts on b, we may write {Y} as the sum of two terms

$$\{Y\} = \{Y\}_1 + \{Y\}_2\ ,$$

where

$$\{Y\}_1 = -\frac{4\pi e^2 \hat{n}_b}{m_b c^2} \int db \, \frac{\omega_c r_b^2}{\gamma} \sum_m J_m'(\lambda_{n\ell}' r_b) \; J_m(\lambda_{n'\ell}' r_b)\left(\frac{\omega}{\omega_c} - \frac{m}{\gamma}\right)$$

$$\cdot \; \delta(b) \frac{\partial}{\partial b} [b \; J_{\ell+m}(\lambda_{n\ell}' b) \; J_m'(\lambda_{n'\ell}' b)] \frac{1}{m \frac{\omega_c}{\gamma} - \omega} \quad , \tag{3.48}$$

$$\{Y\}_2 = \frac{4\pi e^2 \hat{n}_b}{m_b c^2} \frac{\omega_c r_b^2}{\gamma} \int db \, \frac{1}{b} \, \delta(b) \sum_m J_m'(\lambda_{n\ell}' r_b) \; J_m'(\lambda_{n'\ell}' r_b) \; \frac{(\ell+m) r_b}{\gamma}$$

$$\cdot \; \frac{\partial}{\partial b} [J_{\ell+m}(\lambda_{n\ell}' b) \; J_{\ell+m}(\lambda_{n'\ell}' b)] \frac{1}{m \frac{\omega_c}{\gamma} - \omega} \quad . \tag{3.49}$$

In Appendix B we derive the formulae

$$\int_0^R db \, \frac{1}{b} \, \delta(b) \frac{\partial}{\partial b} [b J_{\ell+m}(\lambda_{n\ell}' b) \; J_{\ell+m}'(\lambda_{n'\ell}' b)]$$

$$= \frac{\lambda_{n\ell}'}{4} (\delta_{m,-\ell+1} + \delta_{m,-\ell-1}) - \frac{\lambda_{n'\ell}'}{2} \delta_{m,-\ell} \; , \tag{3.50}$$

$$\int_0^R db \, \frac{1}{b} \, \delta(b) \frac{\partial}{\partial b} [J_{\ell+m}(\lambda_{n\ell}' b) \; J_{\ell+m}(\lambda_{n'\ell}' b)]$$

$$= \frac{1}{4} \lambda_{n\ell}' \lambda_{n'\ell}' (\delta_{m,-\ell+1} + \delta_{m,-\ell-1}) - \frac{1}{4} (\lambda_{n\ell}'^2 + \lambda_{n'\ell}'^2) \; \delta_{m,-\ell} \; . \tag{3.51}$$

With the aid of (3.50) Eq. (3.48) becomes

$$\{Y\}_1 = \frac{4\pi e^2 \hat{n}_b}{\gamma m_b c^2} r_b^2 \left\{\frac{\lambda'_{n\ell}}{2} [J'_{\ell-1}(\lambda'_{n\ell} r_b)\ J_{\ell-1}(\lambda'_{n'\ell} r_b)\right.$$

$$\left. + J'_{\ell+1}(\lambda'_{n\ell} r_b)\ J_{\ell+1}(\lambda'_{n'\ell} r_b)] - \lambda'_{n'\ell}\ J'_\ell(\lambda'_{n\ell} r_b)\ J_\ell(\lambda'_{n'\ell} r_b)\right\}$$

and using (3.51) Eq. (3.49) becomes

$$\{Y_2\} = -\frac{4\pi e^2 \hat{n}_b}{\gamma m_b c^2} \frac{\omega_c r_b^2}{\gamma} \frac{\lambda'_{n\ell}\lambda'_{n'\ell}}{2}$$

$$\cdot \left[\frac{J'_{\ell-1}(\lambda'_{n\ell} r_b) J'_{\ell-1}(\lambda'_{n'\ell} r_b)}{(\ell-1)\ \frac{\omega_c}{\gamma} + \omega} - \frac{J'_{\ell+1}(\lambda'_{n\ell} r_b) J'_{\ell+1}(\lambda'_{n'\ell} r_b)}{(\ell+1)\ \frac{\omega_c}{\gamma} + \omega}\right] .$$

Combining these results we have for the susceptibility of the E-layer electrons at radius r_b:

$$y^{\alpha\alpha}_{nn'}(\omega, r_b) = -\frac{4\pi e^2 \hat{n}_b}{\gamma m_b c^2} \frac{\omega}{\ell\ \frac{\omega_c}{\gamma} + \omega} \frac{\partial}{\partial r_b} [r_b^2\ J'_\ell(\lambda'_{n\ell} r_b)\ J'_\ell(\lambda'_{n'\ell} r_b)]$$

$$+ \frac{4\pi e^2 \hat{n}_b}{\gamma m_b c^2} \frac{\omega^2}{c^2} r_b\ J'_\ell(\lambda'_{n\ell} r_b)\ J'_\ell(\lambda'_{n'\ell} r_b)\ \frac{\omega_c^2 r_b^2}{\gamma^2} \frac{1}{(\ell\ \frac{\omega_c}{\gamma} + \omega)^2} + \frac{4\pi e^2 \hat{n}_b}{\gamma m_b c^2} r_b^2$$

$$\cdot \left\{\frac{\lambda'_{n\ell}}{2} [J'_{\ell-1}(\lambda'_{n\ell} r_b)\ J_{\ell-1}(\lambda'_{n'\ell} r_b) + J'_{\ell+1}(\lambda'_{n\ell} r_b)\ J_{\ell+1}(\lambda'_{n'\ell} r_b)]\right.$$

$$\left. - \lambda'_{n'\ell}\ J'_\ell(\lambda'_{n\ell} r_b)\ J_\ell(\lambda'_{n'\ell} r_b)\right\} - \frac{4\pi e^2 \hat{n}_b}{\gamma m_b c^2} \frac{\omega_c r_b^3}{\gamma} \frac{\lambda'_{n\ell}\lambda'_{n'\ell}}{2}$$

$$\cdot \left[\frac{J'_{\ell-1}(\lambda'_{n\ell} r_b) J_{\ell-1}(\lambda'_{n'\ell} r_b)}{(\ell-1)\ \frac{\omega_c}{\gamma} + \omega} - \frac{J'_{\ell+1}(\lambda'_{n\ell} r_b) J'_{\ell+1}(\lambda'_{n'\ell} r_b)}{(\ell+1)\ \frac{\omega_c}{\gamma} + \omega}\right] . \quad (3.52)$$

The other matrix elements are obtained through very similar calculations, and are given below:

$$y^{\alpha\beta}_{nn'}(\omega,r_b) = \frac{4\pi e^2\hat{n}_b}{\gamma\lambda_{n'\ell}\omega m_b}\left(\frac{\omega^2}{c^2}-\lambda^2_{n'\ell}\right)\frac{\ell}{\ell\frac{\omega_c}{\gamma}+\omega}\frac{\partial}{\partial r_b}\left[r_b J'_\ell(\lambda'_{n\ell}r_b)\, J_\ell(\lambda_{n'\ell}r_b)\right]$$

$$-\frac{4\pi e^2\hat{n}_b}{\gamma m_b}\left(\frac{\omega^2}{c^2}-\lambda^2_{n'\ell}\right)r_b\, J'_\ell(\lambda'_{n\ell}r_b)\, J_\ell(\lambda_{n'\ell}r_b)\frac{1}{\left(\ell\frac{\omega_c}{\gamma}+\omega\right)^2}\quad\frac{\ell\omega_c^2 r_b}{\lambda_{n'\ell}\gamma^3c^2}$$

$$-\frac{4\pi e^2\hat{n}_b}{\omega m_b\gamma}\left(\frac{\omega^2}{c^2}-\lambda^2_{n'\ell}\right)\frac{\lambda'_{n\ell}}{2}\,r_b^2\left[\frac{J'_{\ell-1}(\lambda'_{n\ell}r_b)J_{\ell-1}(\lambda_{n'\ell}r_b)}{(\ell-1)\frac{\omega_c}{\gamma}+\omega}\right.$$

$$\left.-\frac{J'_{\ell+1}(\lambda'_{n\ell}r_b)J_{\ell+1}(\lambda_{n'\ell}r_b)}{(\ell+1)\frac{\omega_c}{\gamma}+\omega}\right]\;, \tag{3.53}$$

$$y^{\beta\alpha}_{nn'}(\omega,r_b) = -\frac{4\pi e^2\hat{n}_b}{m_b\omega_c}\frac{\omega^2}{c^2}\frac{1}{\ell\frac{\omega_c}{\gamma}+\omega}\frac{\partial}{\partial r_b}\left[r_b\, J_\ell(\lambda_{n\ell}r_b)\, J'_\ell(\lambda'_{n'\ell}r_b)\right]$$

$$-\frac{4\pi e^2\hat{n}_b}{\gamma m_b c^2}\frac{\omega^2}{c^2}r_b\, J_\ell(\lambda_{n\ell}r_b)\, J'_\ell(\lambda'_{n'\ell}r_b)\frac{1}{\left(\ell\frac{\omega_c}{\gamma}+\omega\right)^2}\cdot\frac{\ell\omega_c^2 r_b}{\gamma^2}$$

$$+\frac{\omega}{\omega_c}\frac{4\pi e^2\hat{n}_b}{m_b c^2}r_b\left\{\frac{\lambda_{n\ell}}{2}\left[J_{\ell-1}(\lambda_{n\ell}r_b)\, J_{\ell-1}(\lambda'_{n'\ell}r_b)\right.\right.$$

$$\left.\left.+J_{\ell+1}(\lambda_{n\ell}r_b)\, J_{\ell+1}(\lambda'_{n'\ell}r_b)\right]-\lambda'_{n'\ell}\, J_\ell(\lambda_{n\ell}r_b)\, J_\ell(\lambda'_{n'\ell}r_b)\right\}$$

$$-\omega\frac{4\pi e^2\hat{n}_b}{\gamma m_b c^2}\left[\frac{r_bJ_{\ell-1}(\lambda_{n\ell}r_b)J_{\ell-1}(\lambda'_{n'\ell}r_b)}{(\ell-1)\frac{\omega_c}{\gamma}+\omega}-\frac{r_bJ_{\ell+1}(\lambda_{n\ell}r_b)J'_{\ell+1}(\lambda'_{n'\ell}r_b)}{(\ell+1)\frac{\omega_c}{\gamma}+\omega}\right]$$

$$\cdot\frac{\lambda_{n\ell}\lambda'_{n'\ell}r_b}{2}\;, \tag{3.54}$$

$$y^{\beta\beta}_{nn'}(\omega,r_b) = \frac{4\pi e^2\hat{n}_b}{m_b} \frac{1}{\lambda_{n'\ell}} \left(\frac{\omega^2}{c^2} - \lambda^2_{n'\ell}\right) \frac{\ell}{\omega_c} \frac{1}{\ell \frac{\omega_c}{\gamma} + \omega} \frac{\partial}{\partial r_b}$$

$$\cdot [J_\ell(\lambda_{n\ell}r_b)\ J_\ell(\lambda_{n'\ell}r_b)] + \frac{4\pi e^2\hat{n}_b}{\gamma m_b} r_b\left(\frac{\omega^2}{c^2} - \lambda^2_{n'\ell}\right) \frac{1}{\lambda_{n'\ell}} \frac{\ell^2\omega_c^2}{c^2\gamma^2}$$

$$\cdot J_\ell(\lambda_{n\ell}r_b)\ J_\ell(\lambda_{n'\ell}r_b) \frac{1}{(\ell \frac{\omega_c}{\gamma} + \omega)^2} - \frac{4\pi e^2\hat{n}_b}{m_b} \left(\frac{\omega^2}{c^2} - \lambda^2_{n'\ell}\right) \frac{\lambda_{n\ell}r_b}{2\omega_c}$$

$$\cdot \frac{(\ell \frac{\omega_c}{\gamma} + \omega)[J_{\ell-1}(\lambda_{n\ell}r_b)J_{\ell-1}(\lambda_{n'\ell}r_b) - J_{\ell+1}(\lambda_{n\ell}r_b)J_{\ell+1}(\lambda_{n\ell}r_b)]}{(\ell \frac{\omega_c}{\gamma} + \omega)^2 - \frac{\omega_c^2}{\gamma^2}}$$

$$- \frac{4\pi e^2\hat{n}_b}{\gamma m_b} r_b \cdot \frac{\lambda_{n\ell}}{2} \left(\frac{\omega^2}{c^2} - \lambda^2_{n'\ell}\right)$$

$$\cdot \frac{J_{\ell-1}(\lambda_{n\ell}r_b)J_{\ell-1}(\lambda_{n'\ell}r_b) + J_{\ell+1}(\lambda_{n\ell}r_b)J_{\ell+1}(\lambda_{n'\ell}r_b)}{(\ell \frac{\omega_c}{\gamma} + \omega)^2 - \frac{\omega_c^2}{\gamma^2}} \quad . \qquad (3.55)$$

For an E-layer of finite width, with orbits lying between r_{min} and r_{max}, we can find the contribution of the E-layer particles to the total susceptibility by integrating (3.52) - (3.55) over r_b from r_{min} to r_{max}. Thus using (3.19) - (3.20) and (3.52) - (3.55), the elements of the dispersion matrix may be written

$$D^{\alpha\alpha}_{nn'}(\omega) = (\frac{\omega^2}{c^2} - \lambda'^2_{n\ell}) \frac{R^2}{2} [(1 - (\frac{\ell}{\lambda'_{n\ell}R})^2] J^2_\ell(\lambda'_{n\ell}R) \delta_{nn'}$$

$$+ \chi^{\alpha\alpha}_{nn'}(\omega) + \int_{r_{min}}^{r_{max}} dr_b\, y^{\alpha\alpha}_{nn'}(\omega,r_b) ,$$

$$D^{\alpha\beta}_{nn'}(\omega) = \chi^{\alpha\beta}_{nn'}(\omega) + \int_{r_{min}}^{r_{max}} dr_b\, y^{\alpha\beta}_{nn'}(\omega,r_b) ,$$

$$D^{\beta\alpha}_{nn'}(\omega) = \chi^{\beta\alpha}_{nn'}(\omega) + \int_{r_{min}}^{r_{max}} dr_b\, y^{\beta\alpha}_{nn'}(\omega,r_b) , \tag{3.56}$$

$$D^{\beta\beta}_{nn'}(\omega) = (\frac{\omega^2}{c^2} - \lambda^2_{n\ell}) \lambda_{n\ell} \frac{R^2}{2} J^2_{\ell+1}(\lambda_{n\ell}R) \delta_{nn'} + \chi^{\beta\beta}_{nn'}(\omega)$$

$$+ \int_{r_{min}}^{r_{max}} dr_b\, y^{\beta\beta}_{nn'}(\omega,r_b) .$$

The unstable roots are then obtained by truncating the dispersion matrix for some suitable range of n and solving the resulting dispersion relation:

$$\det[D(\omega)] = 0 . \tag{3.57}$$

We will next illustrate this procedure with two numerical examples. For the first, in order to compare our results with those of Striffler and Kammash, we consider parameters appropriate to the Astron experiment. The dimensions in Fig. (3.1) are then r_{min} = 30 cm, r_{max} = 50 cm, t_{EL} = 20 cm, R = 70 cm, and B ≅ 381 Gauss. For these calculations we take the temperature of the background plasma to be one eV and consider a range of plasma densities with the parameter $\frac{\omega_{PP}}{\omega_{CP}}$ ranging from .1 to .9. We take the E-layer to be "cold" in the sense that the gyrocenters of all the E-layer particles are located at the origin. Thus unlike the uniform monoenergetic beam treated by Striffler and Kammash, in the actual cylindrical geometry the value of the relativistic factor γ, and thus of the cyclotron frequency ω_{CB}, will vary across the width of the E-layer. The value of γ was taken to be 9.0 in the calculations of Striffler and Kammash, so we take $\gamma = 9.0$ in the middle of the E-layer, which gives

$$\omega_{CB} = \frac{\omega_{CP}}{\gamma} = 7.45 \times 10^8 \text{ s}^{-1} \quad \text{at} \quad r = 40 \text{ cm} .$$

The particle density of the E-layer is taken, as in Striffler and Kammash, to be given by $\frac{\omega_{CP}}{\omega_{CB}} = .3$.

It now remains to decide how to truncate the D-matrix, and then to evaluate numerically the terms in (3.56) and (3.57). An appropriate truncation of D may be determined by trial and error: that is, we choose an $N \times N$ submatrix D_N of D, find ω_o that satisfies $\det[D_N(\omega_o)] = 0$, and look at the eigenvector of $D_N(\omega_o)$ corresponding to the zero eigenvalue. If the coefficients α_n, β_n of this eigenvector are negligible for $n \cong N$ compared to the coefficients with small n, we may tentatively conclude that we have taken a large enough range of values of n to represent this mode accurately. This is not a rigorous conclusion, however, since there is the possibility that for some values of $n > N$ the coefficients α_n and β_n might be significantly large. If this behavior is suspected for a mode, it can be checked for by truncating the dispersion matrix at an N so large (about 100 in the present problem) that the local uniform approximation used by Striffler and Kammash is valid for $n > N$. The D-matrix is diagonal in n in the local approximation, so that solutions (both frequencies and eigenvectors) of an $N \times N$ submatrix dispersion relation will also be solutions of any higher order truncation. This is usually not very feasible in practice, however, since the larger matrices involved require much larger amounts of computer time and storage.

For the problem considered here, a 20×20 truncation of the dispersion matrix was used; we shall see that this size appears to be more than adequate for most of the modes we find. An example for which a much larger truncation of the dispersion matrix must be used will be discussed later.

To evaluate the elements of the dispersion matrix, we must perform the integration over r_b in (3.56) numerically. Looking at (3.52) - (3.55), we see that if γ were independent of r_b, we could take the resonant denominators outside the integrals, and thus make the integrands independent of ω. Thus in evaluating the dispersion matrix for several values of ω (as we must to find the roots of the dispersion relation), we would have to perform the numerical integration only once, saving a great amount of work. Since actually

γ varies through a range of ±25% around the r = 40 value of 9.0 (it is essentially proportional to r_b), this simplification would give rather crude results. However, we can subdivide the E-layer into nested sections, across each one of which γ may be regarded as constant, and perform the integrals of (3.56) for each of these subdivisions separately. Since γ varies by only 25% across the width of the E-layer and its variation is smooth (essentially linear), only a few of these subdivisions are necessary compared to the many required to accurately integrate the rapidly varying products of Bessel functions which form the remainder of the integrand. Thus in the numerical calculation of most of the modes, γ was considered to vary in ten steps across the E-layer, while 100 subdivisions were used for the rest of the integrand. Simpson's rule was used to perform the integration. For purposes of comparison, as described below, some calculations were also done with γ constant across the E-layer and with γ varying in 20 steps across the layer.

Two approaches were used for finding the roots of the dispersion relation. In the preliminary calculations, for simplicity γ was regarded as constant across the E-layer. It was desired to find all the roots in the vicinity of the upper hybrid frequency, which can be accomplished by determining the number of roots in this region by contour integration. If we have a region R of the complex plane bounded by a closed regular curve S, it is a well-known result[15] that for any analytic function f(z) if N is the number of roots of $f(z)$ (counting multiplicites) in R and P is the number of poles in R (again counting multiplicites) then

$$N - P = \frac{1}{2\pi i} \oint_S \frac{f'(z)}{f(z)} dz \ . \tag{3.58}$$

From (3.52) - (3.55), we see that the only poles of $\det(D(\omega))$ occur for real values of ω; since we are interested only in unstable roots, we take the region R to lie entirely in the upper half of the complex plane. Writing $G(\omega) = \det[D(\omega)]$, we have

$$\begin{aligned} N &= \frac{1}{2\pi i} \oint_S \frac{G'(\omega)}{G(\omega)} d\omega = \frac{1}{2\pi i} \oint_S \frac{d}{d\omega} [\ln G(\omega)] \, d\omega \\ &= \frac{1}{2\pi i} \Delta \ln G(\omega) = \frac{1}{2\pi} \Delta\Theta \end{aligned} \tag{3.59}$$

where Δ is an operator representing the change in a function in going once around S, and $\Delta\Theta$ is the change in the argument of $G(\omega)$ in going once around S. If we represent S by N points s_k on S, $k = 0,1,2,\ldots,N$ with $s_o = s_N$, we have

$$\Delta\Theta = \sum_{k=1}^{N} \tan^{-1} \left[\frac{G(s_{k+1})}{G(s_k)}\right] , \tag{3.60}$$

and (3.60) is <u>exact</u>, provided that the intervals $\{s_k, s_{k+1}\}$ are short enough that

$$\left| \tan^{-1} \frac{G(s_{k+1})}{G(s_k)} \right| < \pi \ .$$

Equation (3.60) simply insures the proper sign for the inverse tangent. The discrete sum (3.60) is of course much easier to use than the continuous integral (3.58).

The initial region R of the ω-plane is shown by the dotted rectangle in Figs. (3.4) - (3.7). After determining the number of roots in this region, using (3.59) and (3.60), the region was repeatedly subdivided until each root was isolated and its position known with sufficient accuracy that it could be found by Newton's method. This procedure was necessary because the roots occur clustered together in the ω-plane, as can be seen in Figs. (3.4) - (3.7), and unless good approximations to the roots are known, Newton's method fails to converge. Only those roots with growth rates greater than $2 \times 10^7 \text{ sec}^{-1}$ were found. There are eight of these in Fig. (3.4), twelve in Fig. (3.5), ten in Fig. (3.6) and eleven in Fig. (3.7). It should be observed that all the roots for a given value of ℓ occur very near resonance with the ℓ^{th} harmonic of the beam cyclotron frequency, $\ell \frac{\omega_c}{\gamma}$.

A similar calculation was carried out for $\ell = 8$, $\frac{\omega_{PP}}{\omega_{CP}} = .1$, and γ varying in ten steps across the E-layer, yielding 19 roots as shown in Fig. (3.8). These roots have a wider spread in their real parts than those for the constant γ calculation; it can be seen that the range of their real parts agrees quite well with the variation in $\frac{\ell\omega_c}{\gamma}$ across the E-layer. We might expect that these modes will be localized in that part of the E-layer for which the ℓ^{th} harmonic of the local beam gyrofrequency $\frac{\ell\omega_c}{\gamma(r)}$ is approximately resonant with the real part of the mode frequency, and we shall see shortly that this is the case. Consequently, only a small part of the E-layer width is able to transfer energy to the mode, and thus we expect the growth rates for these modes to be smaller than for the constant γ calculations in which the entire E-layer is in near resonance with the mode. Comparison of Figs. (3.5) and (3.8) shows that this is true.

The process of finding all the roots of the dispersion relation by contour integration, as in Figs. (3.4) - (3.8), is rather costly in terms of computer time especially when γ is varied across the E-layer. Also, in the varying γ case, we find that the roots are not as tightly bunched in the complex plane as in the constant γ calculation. Thus it turns out that if we scatter some guesses over an area of the complex ω-plane near the appropriate harmonic of the E-layer cyclotron frequency, we find that for a fair fraction of the guesses Newton's method converges. By this method we can obtain a reasonable sample of the roots of the dispersion relation in much less time than it would take to find all of them. For most of the remainder of the calculations then, we will use this approach and take γ to vary in ten steps across the width of the E-layer. We shall also see below that taking 20 steps does not significantly affect the results.

After finding the frequency and growth rate for an unstable mode, it is also of some interest to look at the structure of the mode, particularly its radial dependence. If ω_o is a complex root of the dispersion relation, we can find the eigenvector of $D(\omega_o)$ corresponding to the zero eigenvalue, and this will represent the vector of α_n's and β_n's which when substituted into Eq. (2.14) will yield the vector potential $\underline{A}_1(\underline{r})$ for the mode. From Eq. (2.2) we can then determine the perturbed charge density as

$$\rho_1(\underline{r}) = - \frac{1}{4\pi} (\nabla^2 + \frac{\omega_o^2}{c^2}) \underline{A}_1(\underline{r}) .$$

Figures (3.9) to (3.62) show the results of these calculations for several modes. In each figure is given: the azimuthal mode number ℓ, the value of $\frac{\omega_{PP}}{\omega_{CP}}$, the real frequency and growth rate for the mode, and the absolute values of the coefficients α_n and β_n plotted against n (the squares represent $|\alpha_n|$, the circles $|\beta_n|$). The upper graph in each figure represents $|\rho_1(r)|$ plotted against radius. For most of the modes shown it can be seen that for n near 20, the coefficients α_n and β_n are near zero compared to their values for lower n. This indicates that by truncating the dispersion matrix at n = 20, we have probably kept enough expansion functions to represent the mode quite well. For some of the modes, however, e.g., (3.15), the values of α_n and β_n are still appreciable at n = 20. These modes could either be artifacts of the truncation procedure, and would disappear if a larger truncation of the dispersion matrix were used, or they may represent actual modes which require a larger number of expansion functions for their accurate representation.

To clarify the significance of ρ_1, we compare the perturbed charge density in the background plasma to that in the E-layer, using the results of Striffler and Kammash for the infinite homogeneous case as given in the first part of this chapter. From (3.4), (3.5) and the equation of continuity for charge density:

$$\underline{k} \cdot \underline{J}_1 = \omega\rho_1$$

we have

$$\rho_{1P} = \frac{i\omega_{PP}^2}{4\pi\omega^2} \frac{[\omega^2(\underline{k}\cdot\underline{\varepsilon}) - i\omega\underline{k} \cdot (\underline{\omega}_{CP}\times\underline{\varepsilon})]}{\omega^2 - \omega_{CP}^2} , \tag{3.61}$$

$$\rho_{1B} = \frac{i\omega_{PB}^2}{4\pi\omega^2(\omega-\underline{k}\cdot\underline{V}_B)^2} \tag{3.62}$$

$$\cdot [\omega^2(\underline{k}\cdot\underline{\varepsilon}) + \omega k^2(\underline{V}_B\cdot\underline{\varepsilon}) - \omega(\underline{k}\cdot\underline{V}_B)(\underline{k}\cdot\underline{\varepsilon}) - \frac{\omega}{c^2} (\underline{k}\cdot\underline{V}_B)(\underline{V}_B\cdot\underline{\varepsilon})] .$$

As a rough approximation, we take the terms in square brackets in (3.61) and (3.62) to be the same order of magnitude, and assume also $\omega^2 \cong \omega^2 - \omega_{CP}^2$. Then we have

$$\frac{|\rho_{1B}|}{|\rho_{1P}|} \cong \frac{\omega_{PB}^2}{\omega_{PP}^2} \frac{1}{(\omega-\underline{k}\cdot\underline{V}_B)^2} . \tag{3.63}$$

Since we find that the modes occur near the resonance frequency, we expect $|\rho_{1B}| > |\rho_{1P}|$.

Another way of seeing this is to note that the frequency of the mode is much higher than the background plasma frequency, but in the rest frame of the E-layer it is lower than the E-layer plasma frequency.

From these considerations we expect to find the largest perturbed charge density in the region of the E-layer, between 30 and 50 cm radius. This is seen to be the case in Figs. (3.9) - (3.16), which show some of the modes found for $\frac{\omega_{PP}}{\omega_{CP}} = .1$, $\omega_{PP} = 3\omega_{PB}$, and γ constant across the E-layer. Note that the perturbed charge density for these modes is almost entirely confined to the E-layer, with the boundaries of the E-layer at 30 and 50 cm being quite apparent. The modes are normalized so that the absolute value of the largest expansion coefficient is one, and the charge densities are then calculated from these normalized coefficients. This makes it easier to tell similar modes (such as Figs. (3.14) and (3.16)) apart.

Figures (3.17) - (3.35) show modes for the same plasma parameters but with γ varying in ten steps across the E-layer. Here we see that the perturbed charge density is confined to the region within the E-layer where the mode is resonant with the beam particles. Note that modes of high ω_r are localized at smaller radii, where the E-layer cyclotron frequency is higher. Figures (3.36) and (3.37) show modes corresponding to (3.35) and (3.25), respectively, but calculated with γ varying in 20 steps across the E-layer. The modes, whether calculated with ten or twenty steps of γ, are quite similar in both charge density distribution and expansion coefficients and differ by only a few percent in frequency. Thus the ten-step approximation to γ is seen to be sufficient to yield the modes fairly accurately. The rest of the calculations for this problem will use the ten-step approximation for γ.

Figures (3.38) - (3.60) show the effect of increasing ω_{PP} while holding ω_{PB} constant, i.e., increasing the density of the background plasma while holding the E-layer density constant. This corresponds to the situation in the early stages of the Astron experiment, where the constant density E-layer ionizes more and more of the neutral background. Calculations were done for the same range of plasma densities as those used by Striffler and Kammash. Figures (3.38) - (3.48) have $\frac{\omega_{PP}}{\omega_{CP}} = .3$, so that $\omega_{PP} = 9\omega_{PB}$. From Eq. (3.64) we expect to see the perturbed charge density extend more outside the E-layer, since the background plasma will now make a larger contribution. This effect is evident in the graphs of the charge density, which also show the region of largest $|\rho_1|$ again occurring near resonance between the mode and the $\ell = 8^{th}$ harmonic of the E-layer cyclotron frequency. Note Figs. (3.40) and (3.48) which show stable modes. These modes show a similar relation between the peak of the charge density and the frequency. Figures (3.49) - (3.56) show the same tendencies for modes with $\frac{\omega_{PP}}{\omega_{CP}} = .5$, or $\omega_{PP} = 15\omega_{PB}$. Note Figs. (3.55) and (3.56), which show that two distinct modes may be very close in structure and frequency. For Figs. (3.57) - (3.58), which are typical modes for $\frac{\omega_{PP}}{\omega_{CP}} = .7$ or $\omega_{PP} = 21\omega_{PB}$, and Figs. (3.59) - (3.60), which are typical modes for $\frac{\omega_{PP}}{\omega_{CP}} = .9$, or $\omega_{PP} = 27\ \omega_{PB}$, we see that the density of the background plasma has become so large that its contribution to the perturbed charge density is nearly equal to that of the E-layer.

Similar calculations were performed for other values of the azimuthal mode number ℓ. Modes were found near the ℓ^{th} harmonic of the E-layer cyclotron frequency, and their growth rates and radial structure were quite similar to the $\ell = 8$ case.

Next we compare the results obtained above with those of Striffler and Kammash. Figure (3.61) shows the growth rates obtained by Striffler and Kammash in the local approximation for a variety of ℓ's and n's. In the local approximation the dispersion matrix is diagonal in n, so that each n corresponds to one mode. Since we took only the lowest twenty n's, we are only interested in the left portions of the graphs in Fig. (3.63). The growth rate is plotted versus $k_\perp c/\omega_{CB} = \sqrt{\ell^2 + n^2}$, and we see that the growth rates generally peak near the lowest possible values of $k_\perp c/\omega_{CB}$, i.e., the lowest values of n. We recall from (3.8) that the condition that the local approximation be valid (which followed from the requirement that the radial wavelength be smaller than the thickness of the E-layer) is

$$\frac{k_x c}{\omega_{CB}} = n > 15 .$$

Thus we see that the local approximation is not really valid for those modes which it shows to have the highest growth rate, and to handle these modes correctly we must take into account the finite geometry of the system. And in fact, we have seen that the unstable modes are typically localized in an annulus with about one-tenth the thickness of the E-layer itself so that really we should require $n > 150$ before using the local approximation.

Actually, however, restrictions on n are not sufficient to justify the local approximation, and we cannot expect it to give quantitively correct results for any value of n. This is because it assumes that the entire space is filled uniformly with the E-layer, which corresponds in the actual cylindrical geometry to an E-layer filling the entire cylinder. Also all the particles in the E-layer would have the same cyclotron frequency, regardless of radius, and thus all the particles would be resonant with the unstable mode, and contribute energy to it. Since in the actual situation only those particles in an annular radius of fairly small width are resonant with the mode and contribute energy to it, we expect that the actual growth rates will be much smaller than those calculated by the local approximation. As a crude estimate, we might expect them to be smaller by a factor of $R_{res}/R = .05$, where R_{res} is the width of a typical resonance zone for a mode (about 3 cm from Figs. (3.17) - (3.31)) and R is the radius of the cylinder, 70 cm.

From the graphs in Figs. (3.63), we have an average growth rate of about $.2\omega_{CB} = 1.5 \times 10^8 \text{ sec}^{-1}$ for the $\ell = 8$ modes, and multiplying by .05 gives a growth rate of about $.75 \times 10^7 \text{ sec}^{-1}$, which is the correct order of magnitude, though somewhat smaller than the growth rates we find in the non-local calculation. The difference may be partly due to the fact that in the cylindrical geometry the normal mode of the background Maxwellian plasma, having radial dependence $J_\ell \lambda r)$, will not extend all the way in to $r = 0$. Thus we should multiply the growth rates for the local approximation not by R_{res}/R but by R_{res}/R_{eff}, where $R_{eff} < R$ is the radial extent of the background plasma mode.

Striffler and Kammash do not give numerical results for the real frequencies corresponding to the unstable modes, but we may assume they are quite close to the appropriate harmonic of the E-layer cyclotron frequency, as in our calculations for a constant γ across the E-layer (Figs. 3.4) - 3.7)). Thus the non-local theory predicts a considerably wider spread of real frequencies for the unstable modes of a given azimuthal mode number than the local theory.

To summarize the comparison of local and non-local results: We find that the non-local theory predicts a wider spread of real frequencies and substantially smaller growth rates for the unstable modes corresponding to a given value of ℓ.

The formalism developed in this chapter was also used to calculate some of the unstable modes for the device shown in Fig. (3.62), which is used in experiments described in Ref. (16). Briefly, an annular electron beam with energy 1.2 MV, radius 3 cm, and thickness .5 cm is injected into a plasma-filled chamber. The beam passes through a magnetic cusp in which the axial magnetic field is reversed from $-B_{oz}$ to $+B_{oz}$. This converts most of the electron streaming velocity into rotational velocity. Thus we have a relativistic E-layer in a background plasma, and we may apply the same methods used for the Astron model.

In the experiment, radiation is observed near the upper hybrid frequency. It is believed to be coherent curvature radiation[17] caused by charge bunching in the E-layer. This bunching is believed to be due to instability of the extraordinary electromagnetic mode as in Astron. However, because of the smaller dimensions of the system and especially the fact that t_{EL} = .5 cm is smaller than the wavelength of the radiation, the local approximation used by Striffler and Kammash to treat Astron is not suitable. Thus it is of interest to see if the non-local theory developed here predicts instabilities near the upper hybrid frequency.

The background plasma for this calculation was taken to have a temperature of 1 ev, a plasma frequency of 1.88×10^{11} sec^{-1}, and a cyclotron frequency of 2.46×10^{10} sec^{-1}. Thus its hybrid frequency was 1.90×10^{11} sec^{-1}. The E-layer was taken to have an inner radius of 2.75 cm, an outer radius of 3.25 cm, a density of 1×10^{11} particles/cm^3 ($\cong$ 1% of the background plasma), and a relativistic mass factor $\gamma = 3.3$. These parameters were obtained from the data given in Ref. (16).

Since $\frac{\omega_H}{(\omega_c/\gamma)} = 25.5$, it was decided to look for instabilities of the $\ell = 26$ modes. Initially the calculation was carried out truncating the dispersion matrix at $n = 45$. This yielded the unstable modes shown in Figs. (3.63) - (3.70). The plots of charge density for these modes show the pattern we have come to expect from the Astron model, with the highest peak occurring within or near the E-layer. However, the plots of the expansion coefficients for the modes show that a larger truncation of the dispersion matrix must be taken to obtain an accurate representation of these modes. Two unstable modes calculated with a dispersion matrix truncated at $n = 100$ are shown in Figs. (3.71) and (3.72). The mode in Fig. (3.71) seems to have fairly well converged in n, as indicated by the small coefficients near $n = 100$ and the small peak in charge density at small r. The mode is seen to be quite sharply confined radially to the region of the E-layer. The mode in Fig. (3.72) clearly needs an even larger set of expansion functions. The large peak at $r \cong .8$ is probably an artifact of the truncation, since the mode depends radially on Bessel functions of the form $J_\ell(\lambda_n r)$, which will make contributions to the mode at small r only for large n. Even in this case, however, the main peak of the mode is seen to lie within the

E-layer.

Thus we may conclude that the extraordinary electromagnetic mode will be unstable in this device, and may give rise to charge bunching in the E-layer which will produce the observed radiation.

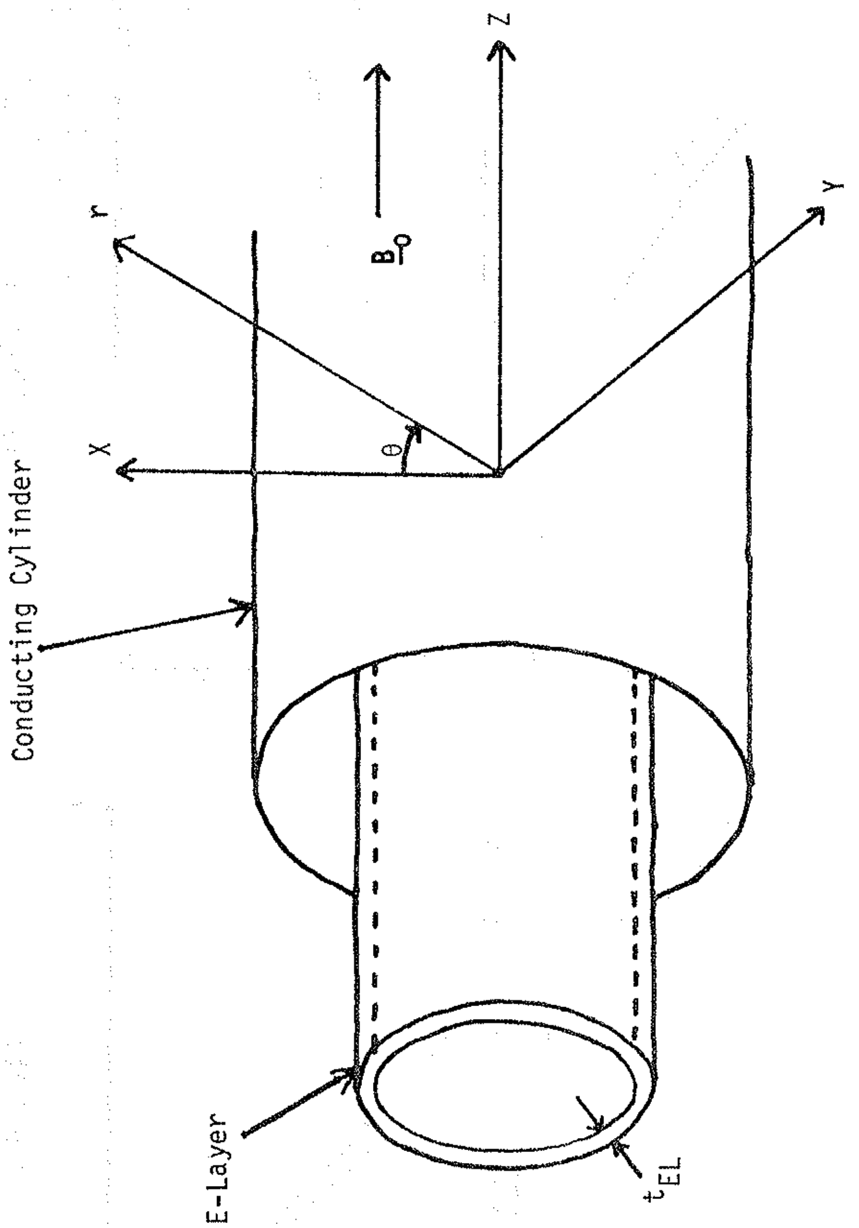
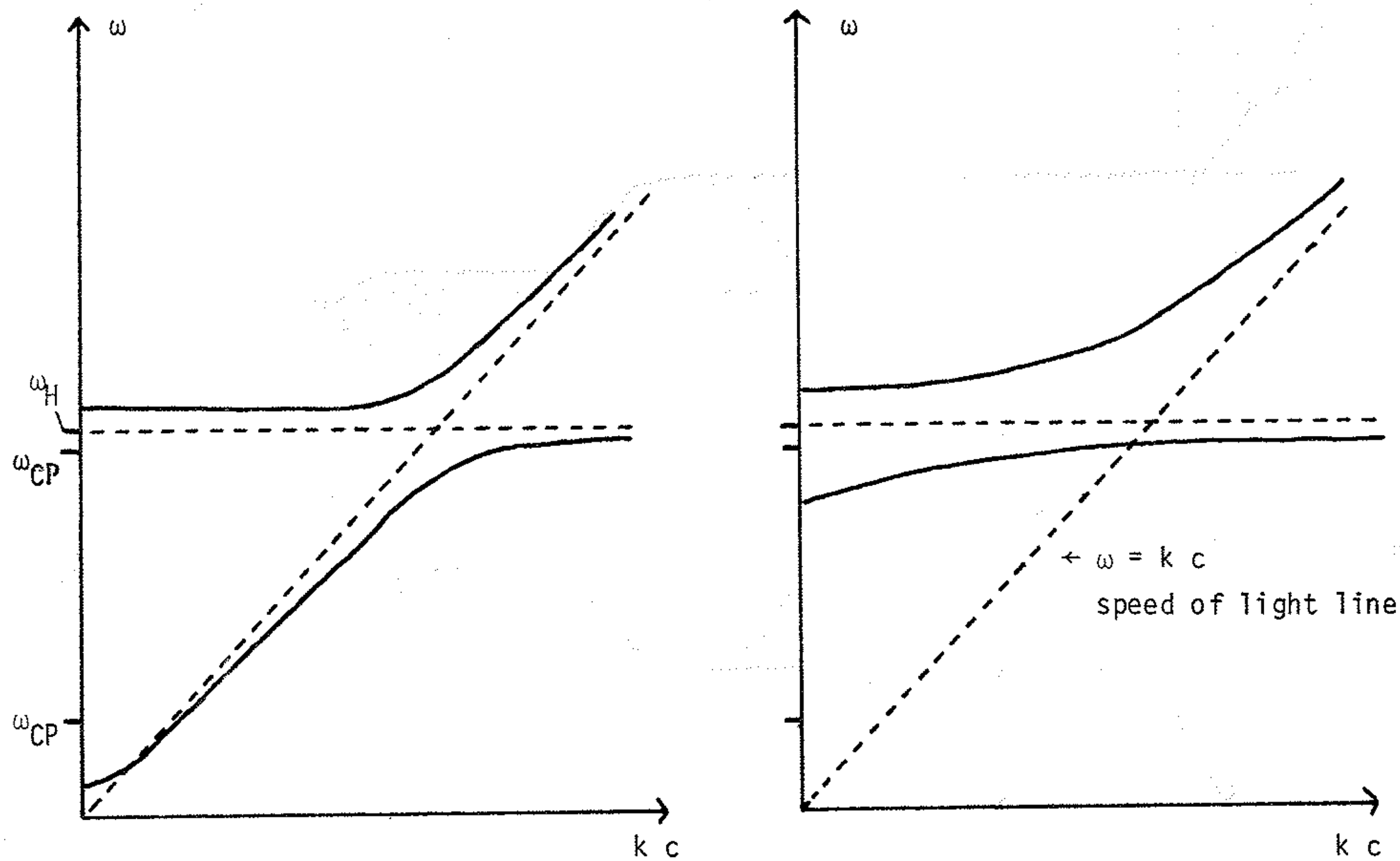


Fig. 3.1



$\omega_{CP} > \omega_{PP}$ $\omega_{PP} > \omega_{CP}$

Fig. 3.2 Extraordinary normal modes of a cold plasma

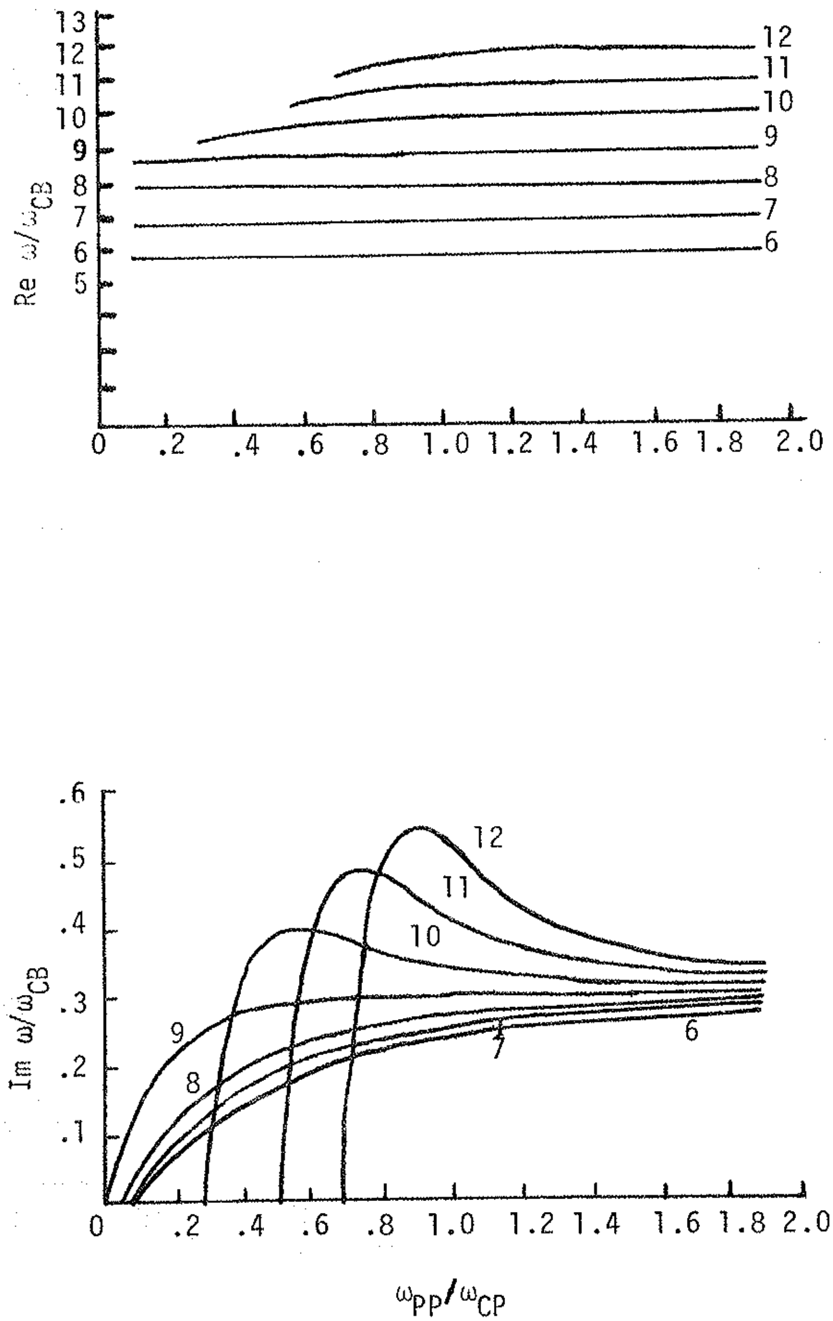


Fig. 3.3 Fluid model frequencies and growth rates

for ℓ = 6-12, large $k_x c/\omega_{CB}$

Fig. 3.4 γ constant, 8 roots

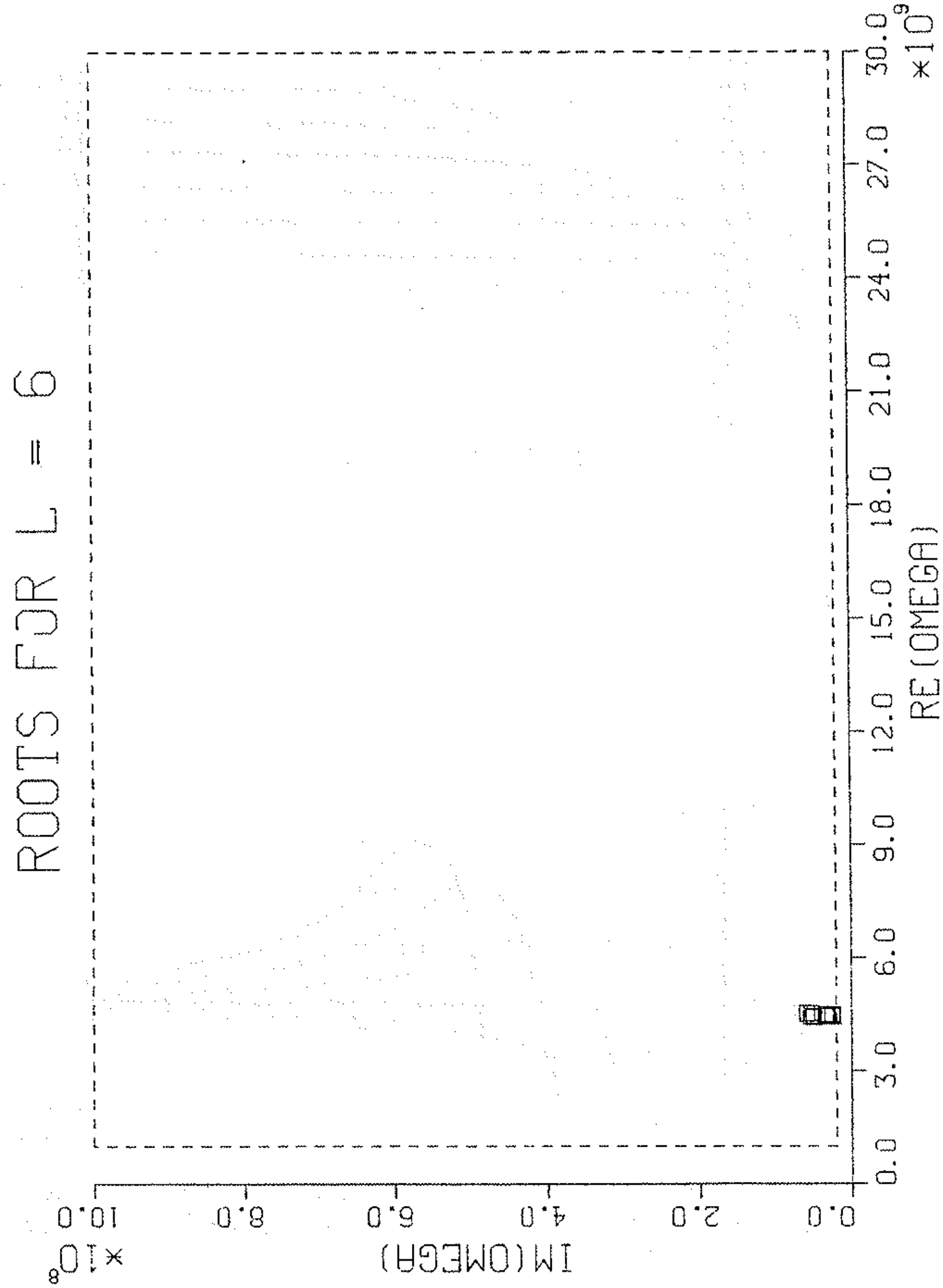


Fig. 3.5 γ constant, 12 roots

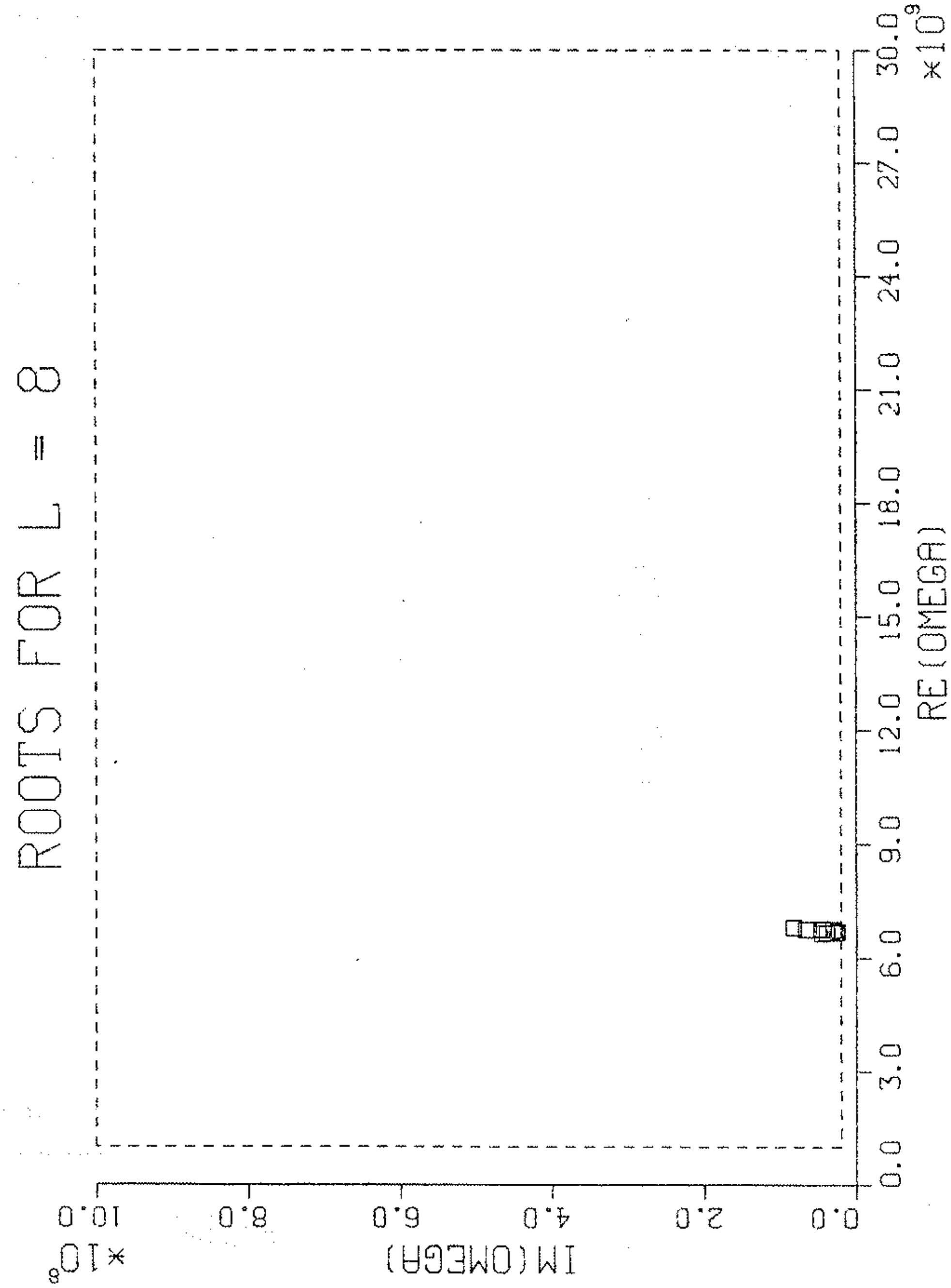


Fig 3.6 γ constant, 10 roots

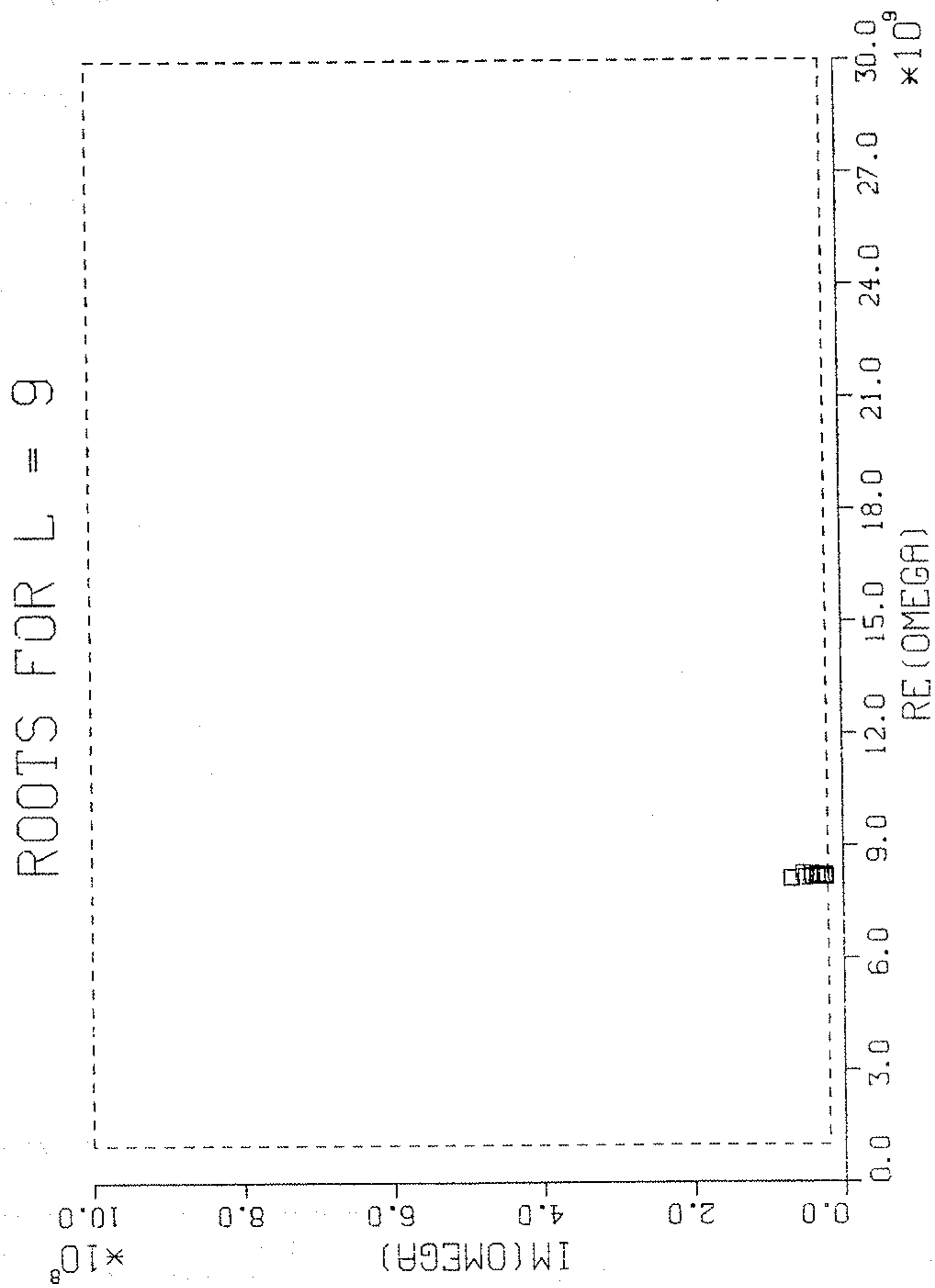


Fig. 3.7 γ constant, 11 roots

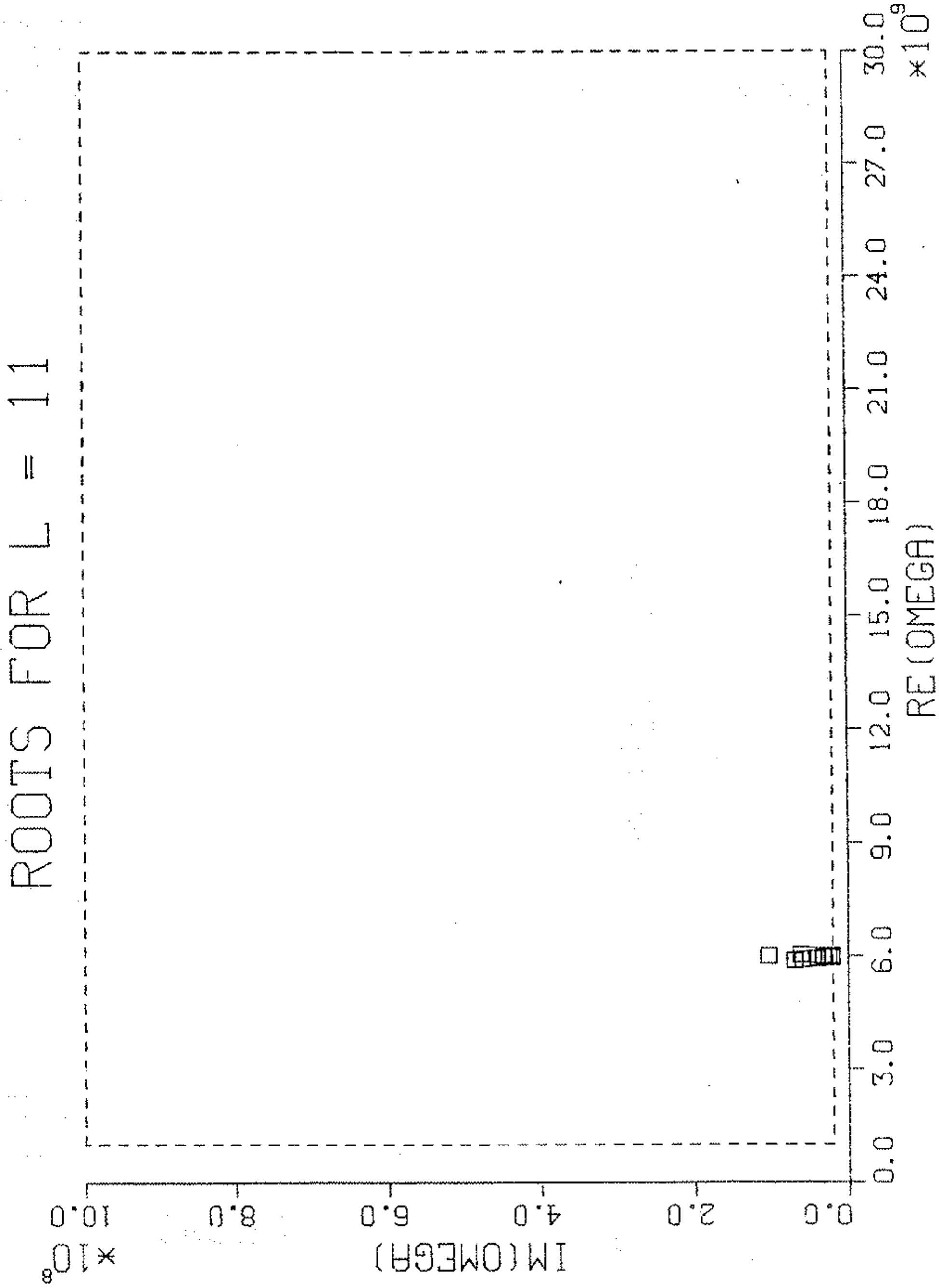


Fig. 3.8 γ - 10 steps, 19 roots

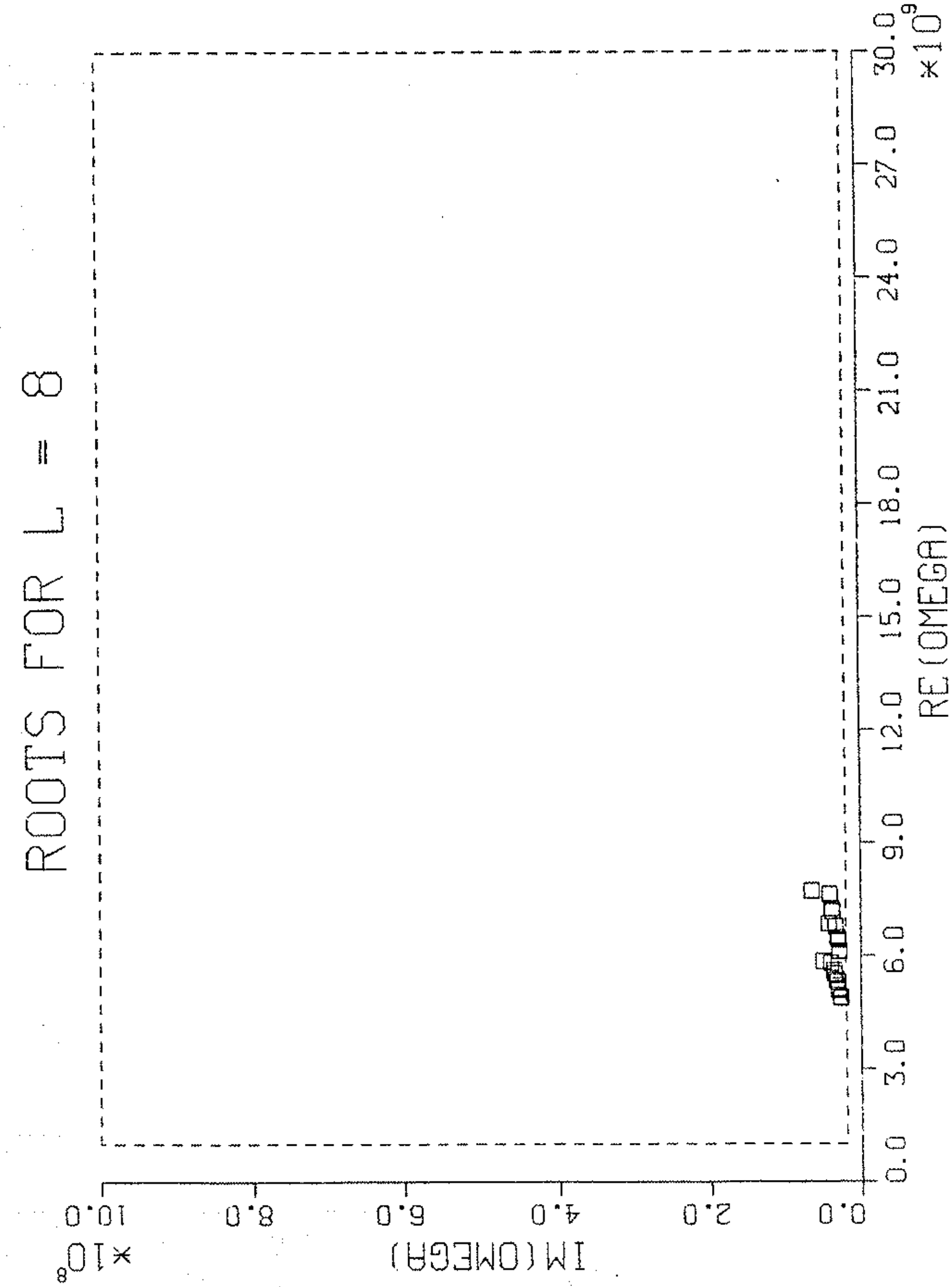
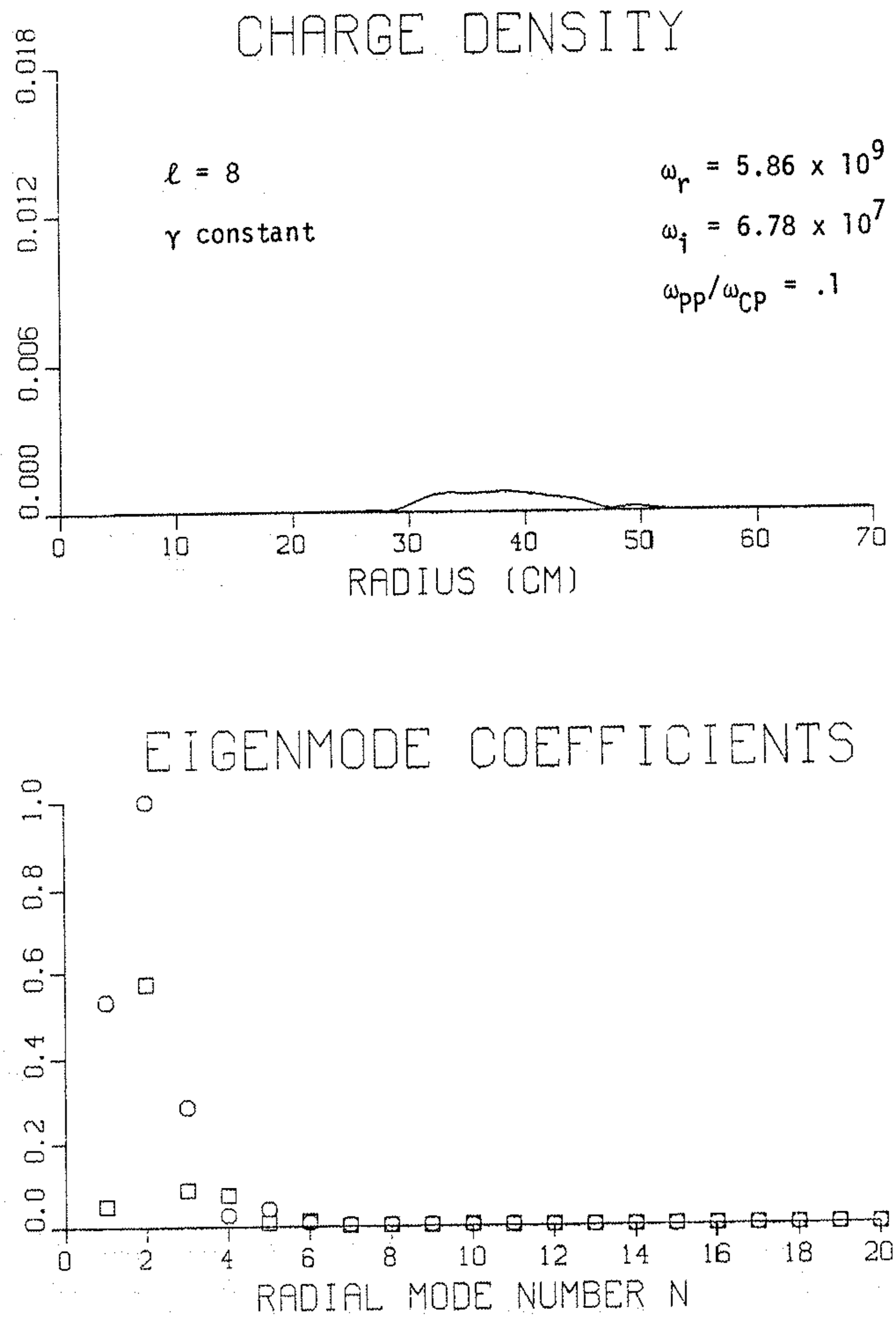


Fig 3.9



CHARGE DENSITY
ℓ = 8
γ constant
ωr = 5.86 x 10^9
ωi = 6.78 x 10^7
ωpp/ωCP = .1
0.000
0.006
0.012
0.018
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig 3.10

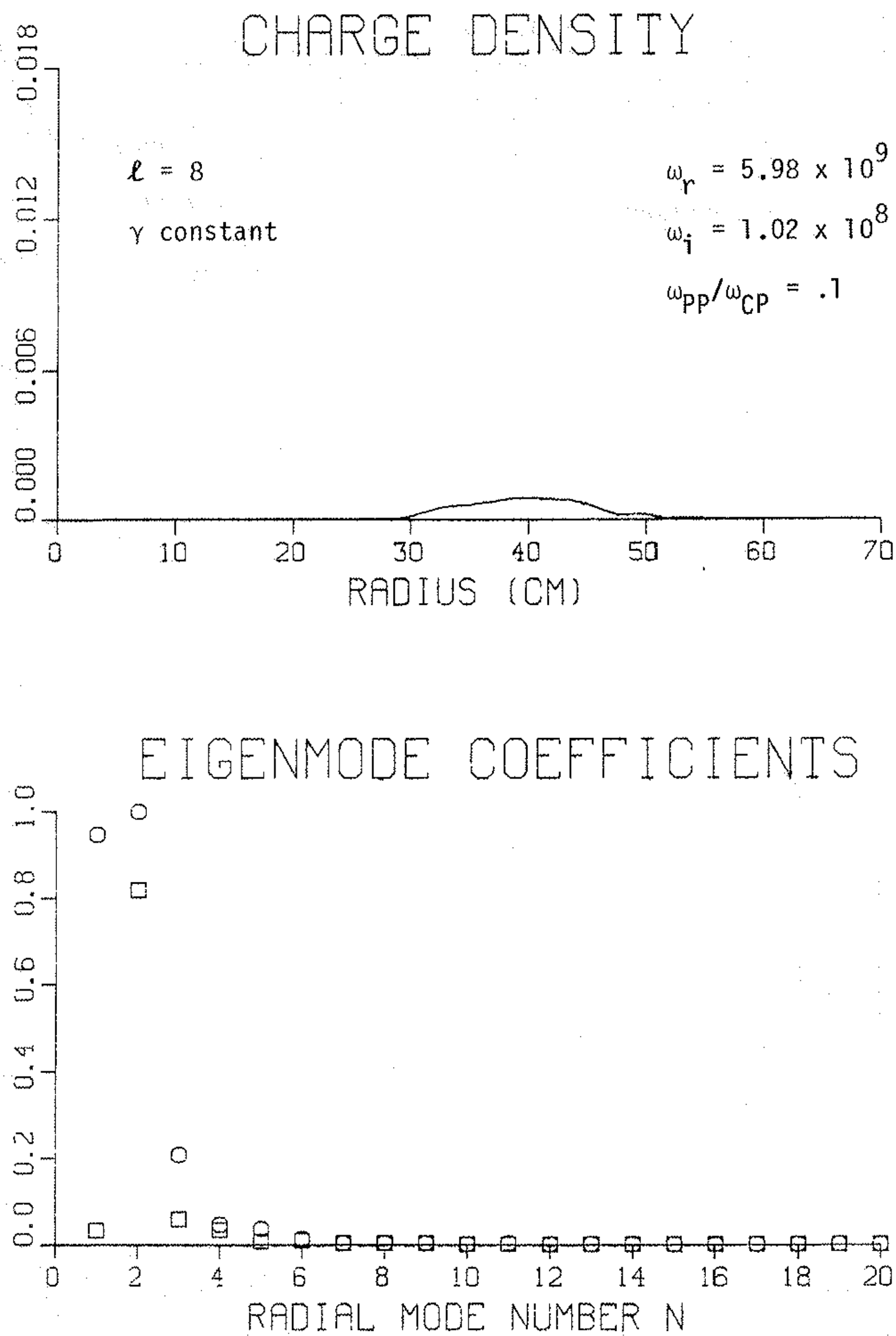
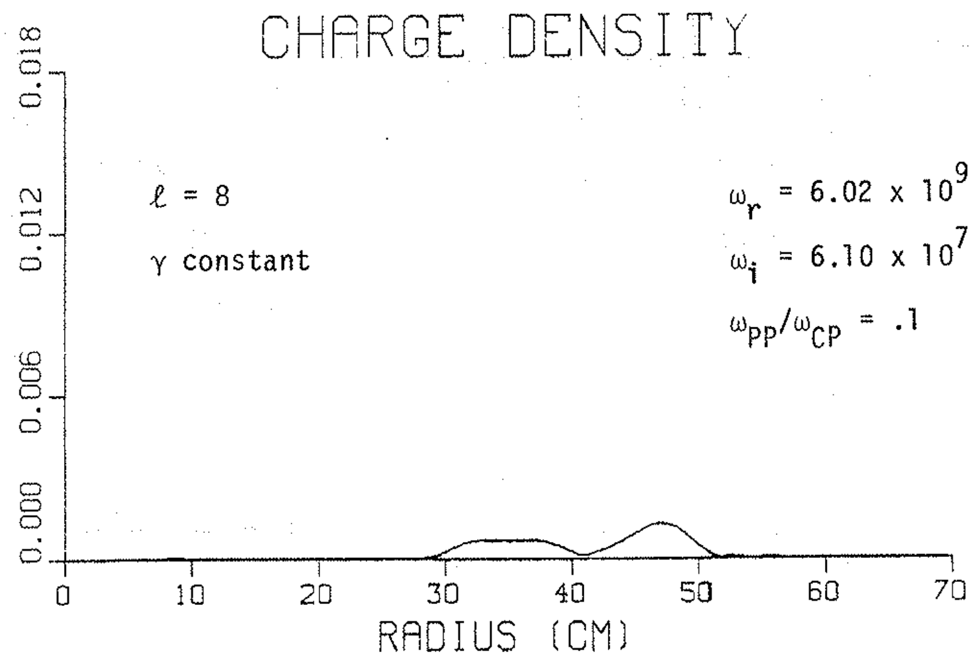

CHARGE DENSITY
ℓ = 8
γ constant
$\omega_r = 5.98 \times 10^9$
$\omega_i = 1.02 \times 10^8$
$\omega_{PP}/\omega_{CP} = .1$
0.018
0.012
0.006
0.000
0 10 20 30 40 50 60 70
RADIUS (CM)
EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0 2 4 6 8 10 12 14 16 18 20
RADIAL MODE NUMBER N

Fig 3.11


CHARGE DENSITY
0.018
0.012
0.006
0.000
ℓ = 8
γ constant
$\omega_r = 6.02 \times 10^9$
$\omega_i = 6.10 \times 10^7$
$\omega_{PP}/\omega_{CP} = .1$
0 10 20 30 40 50 60 70
RADIUS (CM)

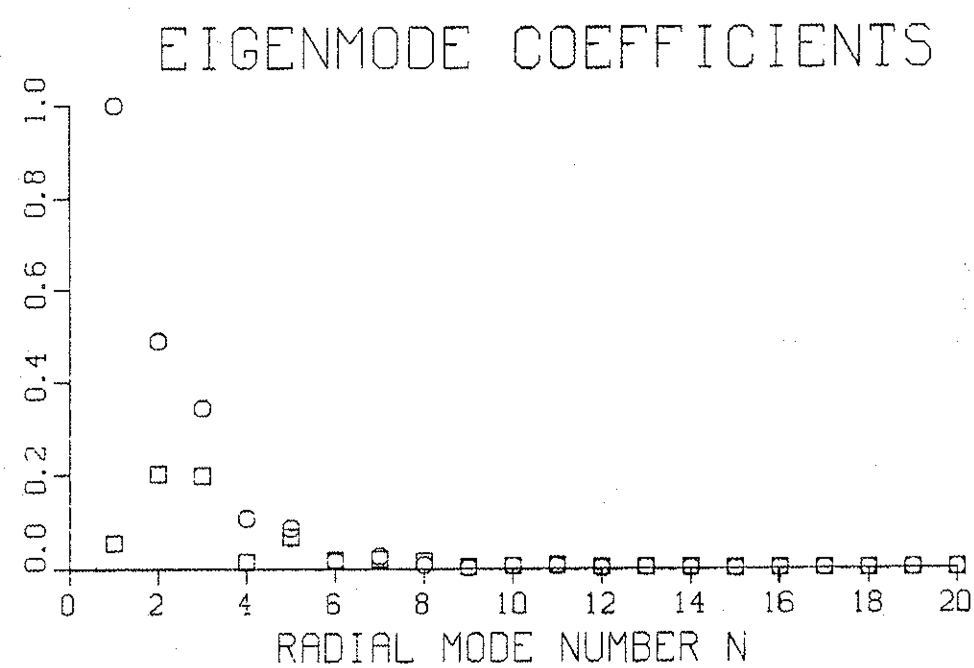

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0 2 4 6 8 10 12 14 16 18 20
RADIAL MODE NUMBER N

Fig 3.12

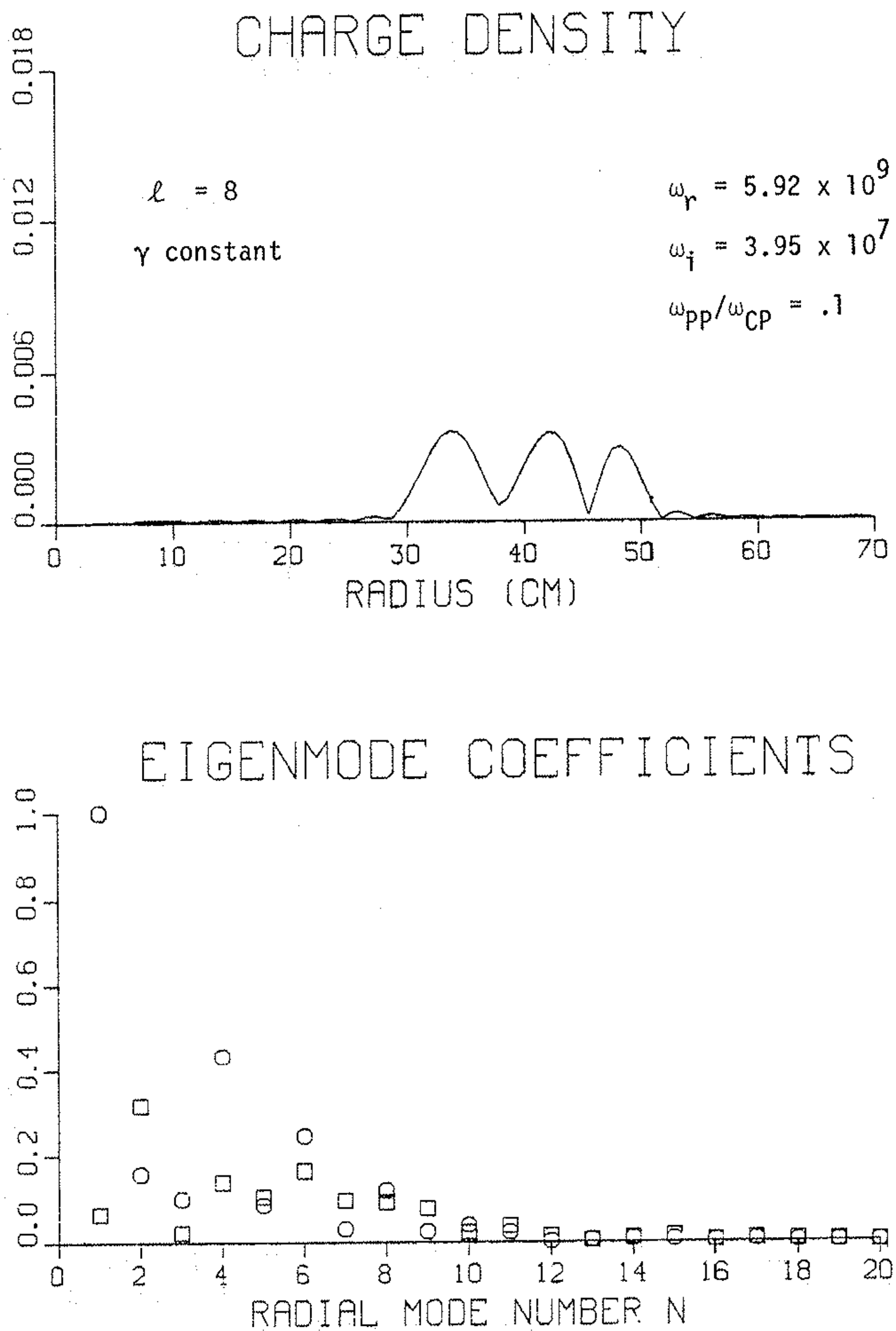

CHARGE DENSITY
0.018
0.012
0.006
0.000
ℓ = 8
γ constant
ω_r = 5.92 x 10^9
ω_i = 3.95 x 10^7
ω_{pp}/ω_{CP} = .1
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig 3.13

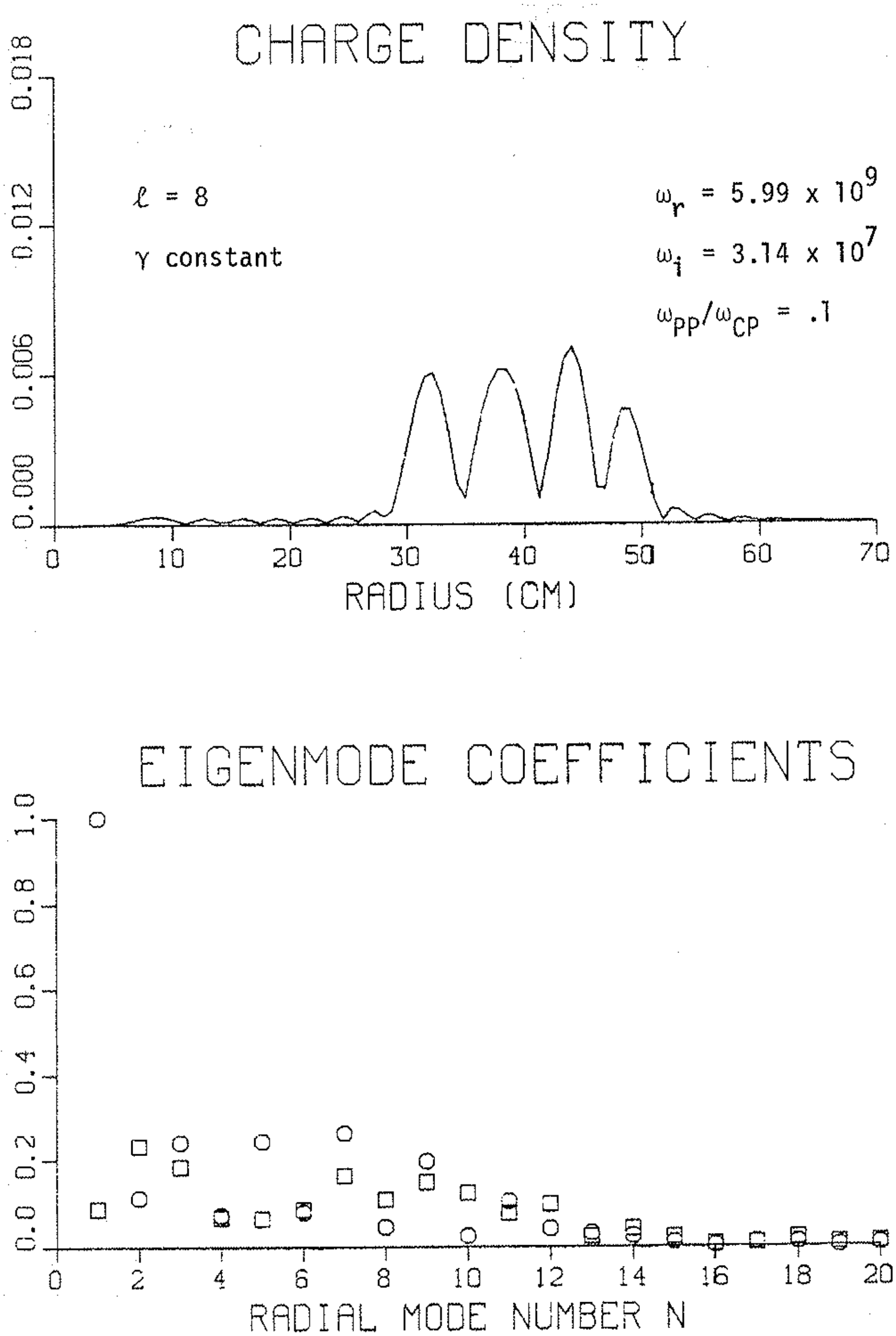
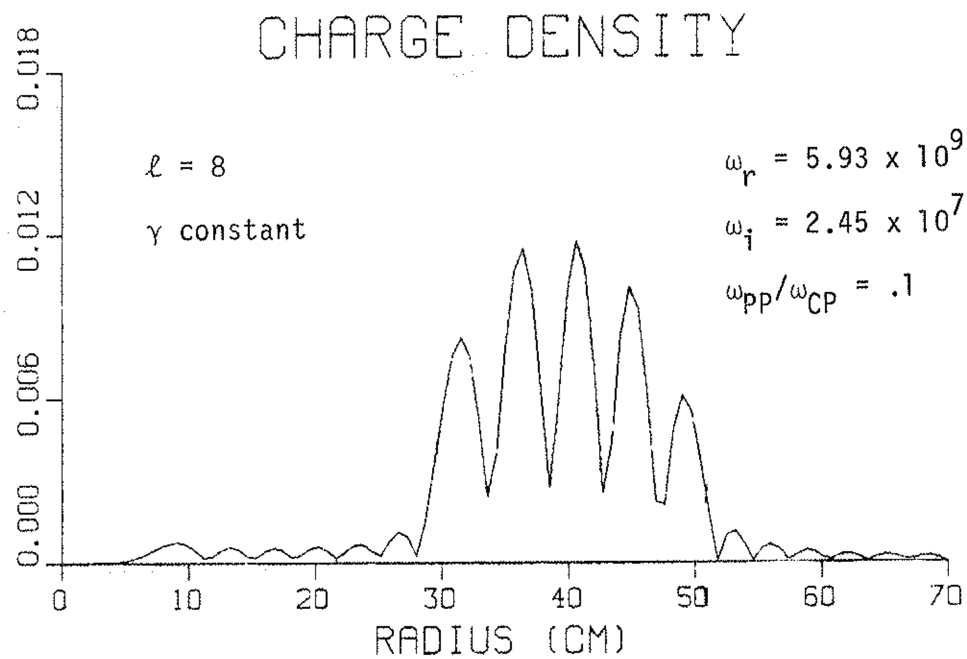

CHARGE DENSITY
ℓ = 8
γ constant
$\omega_r = 5.99 \times 10^9$
$\omega_i = 3.14 \times 10^7$
$\omega_{PP}/\omega_{CP} = .1$
0.000
0.006
0.012
0.018
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig 3.14


CHARGE DENSITY
ℓ = 8
γ constant
ω_r = 5.93 x 10^9
ω_i = 2.45 x 10^7
ω_pp/ω_CP = .1
0.000
0.006
0.012
0.018
0
10
20
30
40
50
60
70
RADIUS (CM)

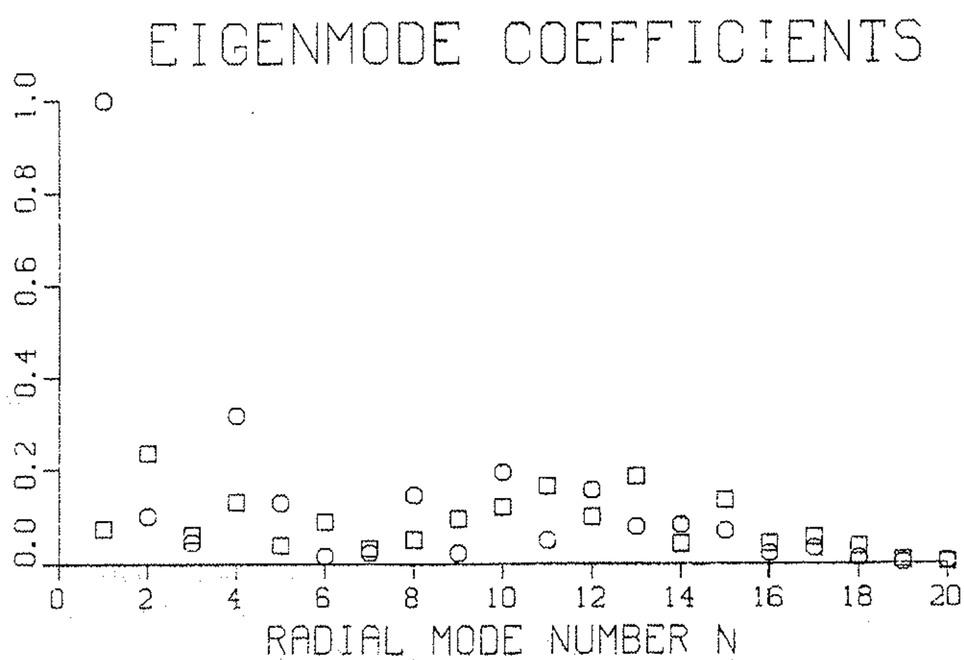
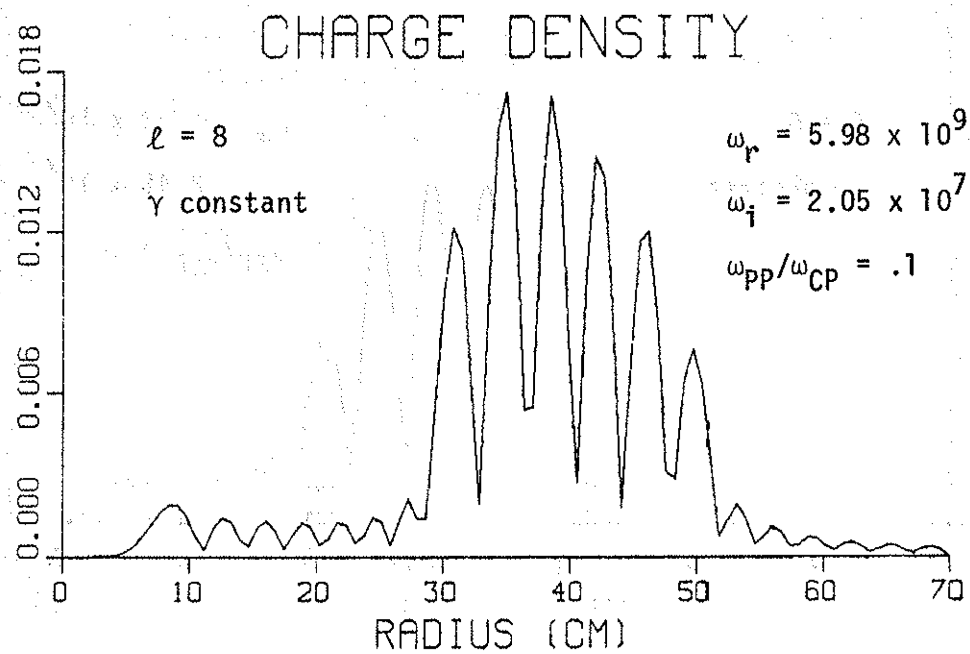

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig 3.15


CHARGE DENSITY
ℓ = 8
γ constant
ωr = 5.98 x 10^9
ωi = 2.05 x 10^7
ωpp/ωcp = .1
0.018
0.012
0.006
0.000
0
10
20
30
40
50
60
70
RADIUS (CM)

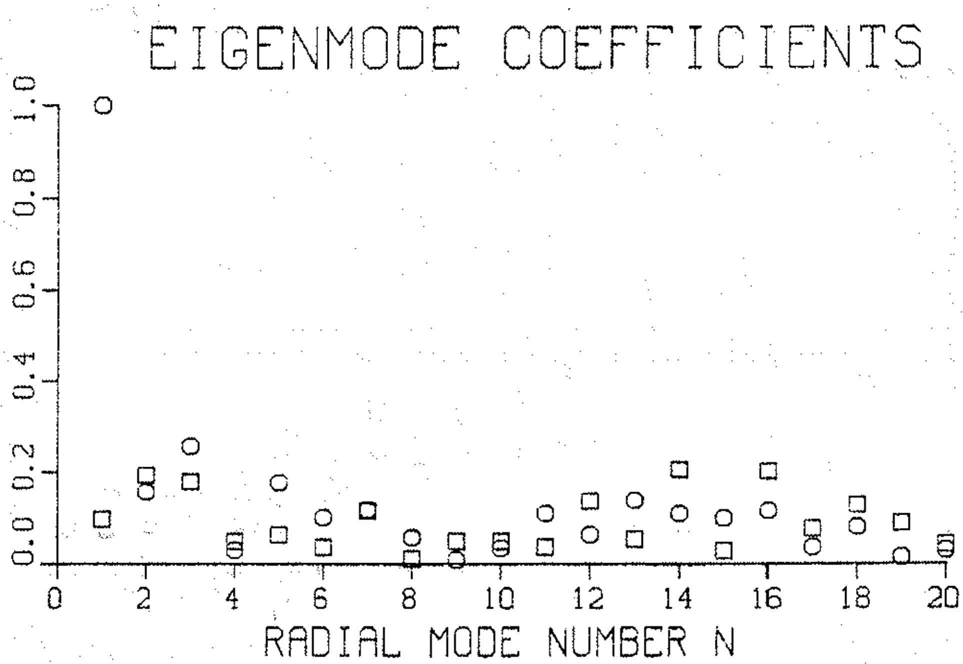
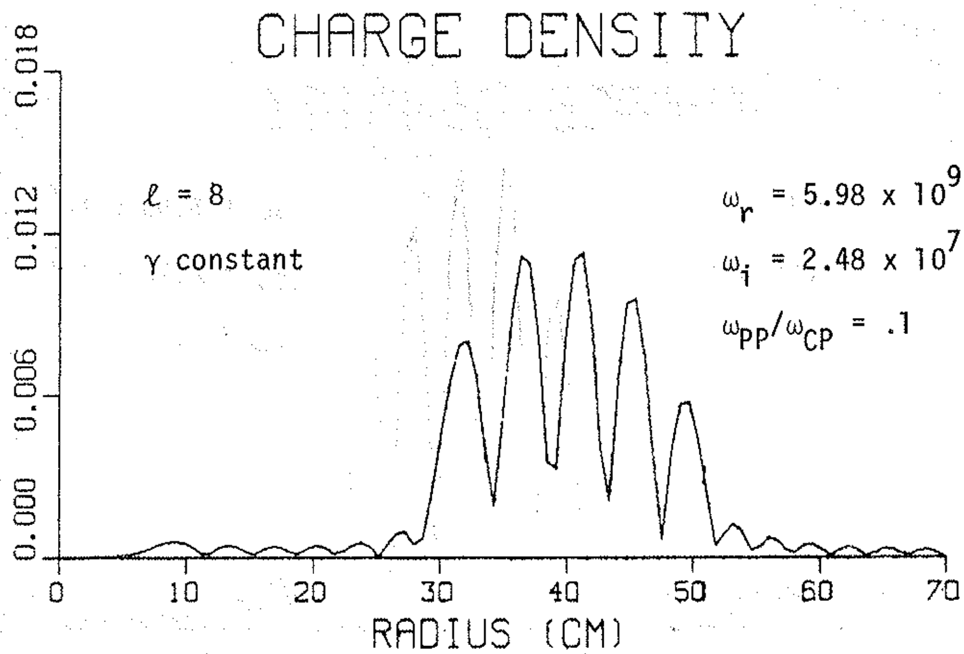

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig 3.16


CHARGE DENSITY
0.000
0.006
0.012
0.018
ℓ = 8
γ constant
ω_r = 5.98 x 10^9
ω_i = 2.48 x 10^7
ω_{PP}/ω_{CP} = .1
0
10
20
30
40
50
60
70
RADIUS (CM)

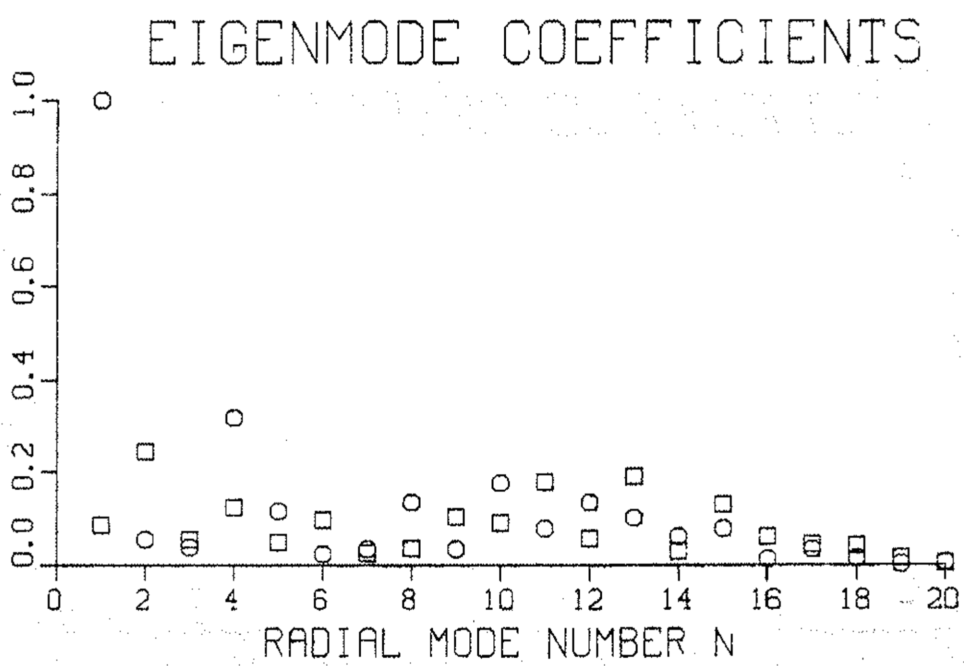
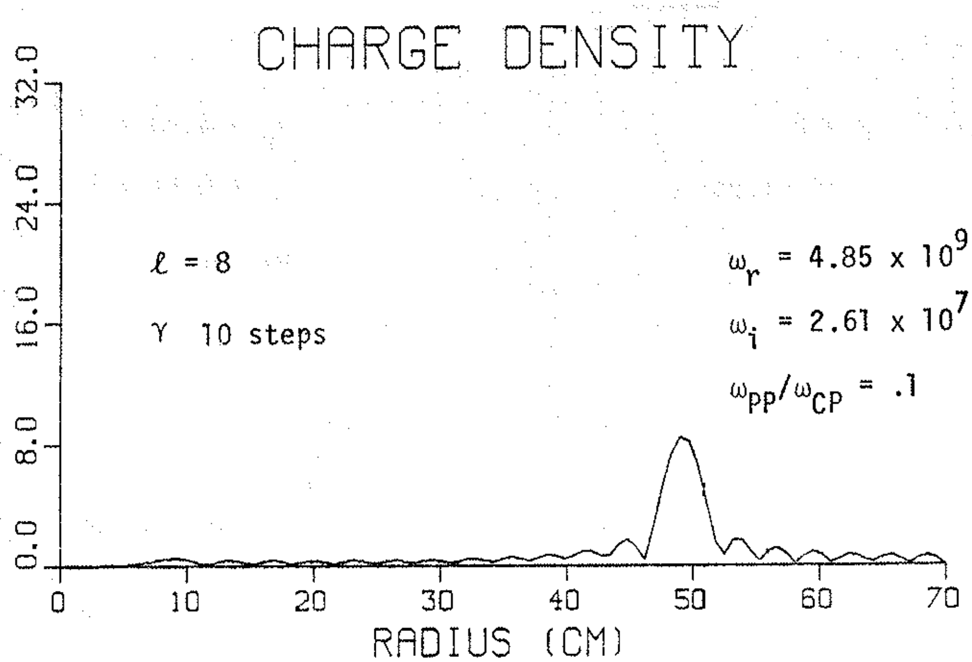

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig 3.17


CHARGE DENSITY
32.0
24.0
16.0
8.0
0.0
ℓ = 8
γ 10 steps
ω_r = 4.85 x 10^9
ω_i = 2.61 x 10^7
ω_{PP}/ω_{CP} = .1
0
10
20
30
40
50
60
70
RADIUS (CM)

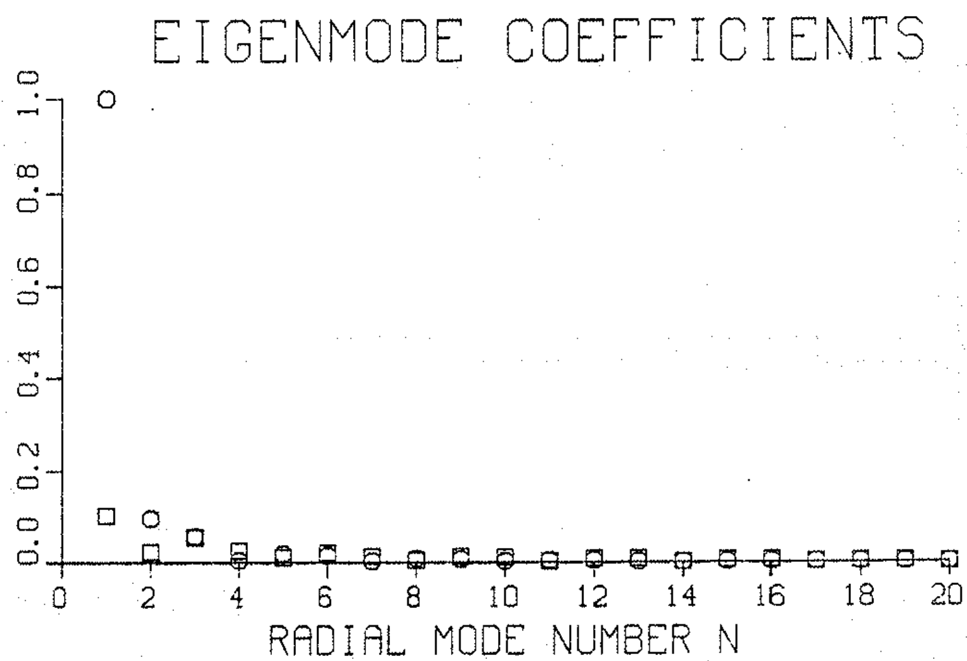
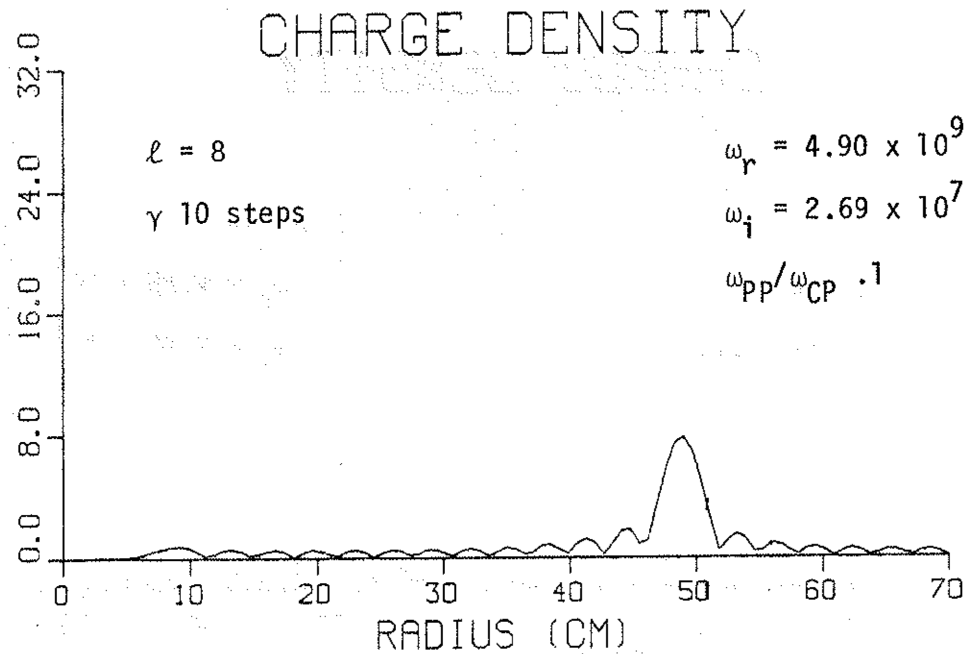

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.18


CHARGE DENSITY
32.0
24.0
16.0
8.0
0.0
ℓ = 8
γ 10 steps
ωr = 4.90 x 10^9
ωi = 2.69 x 10^7
ωpp/ωCP .1
0
10
20
30
40
50
60
70
RADIUS (CM)

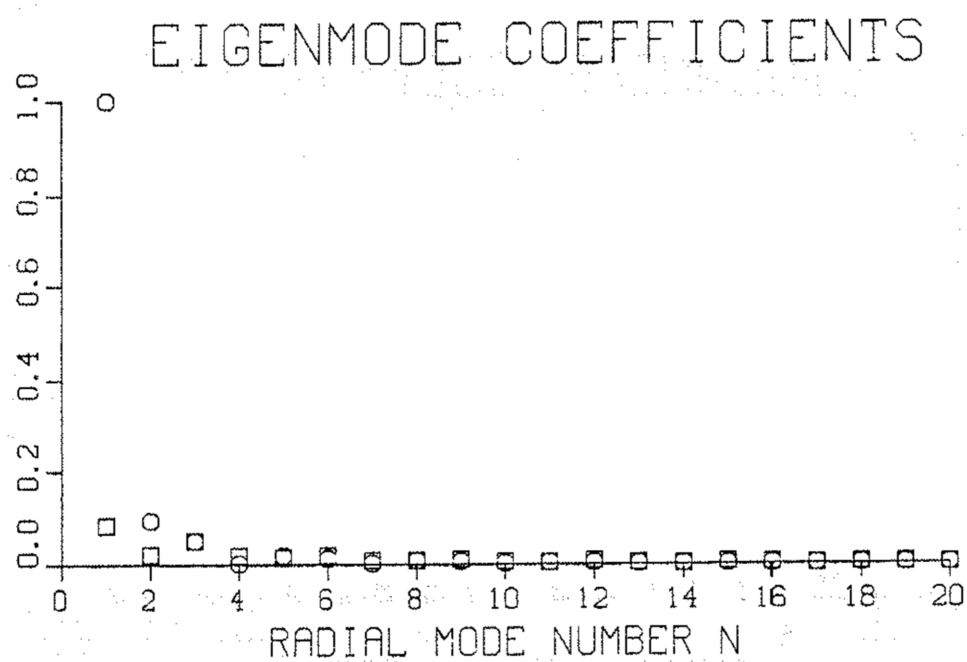

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.19

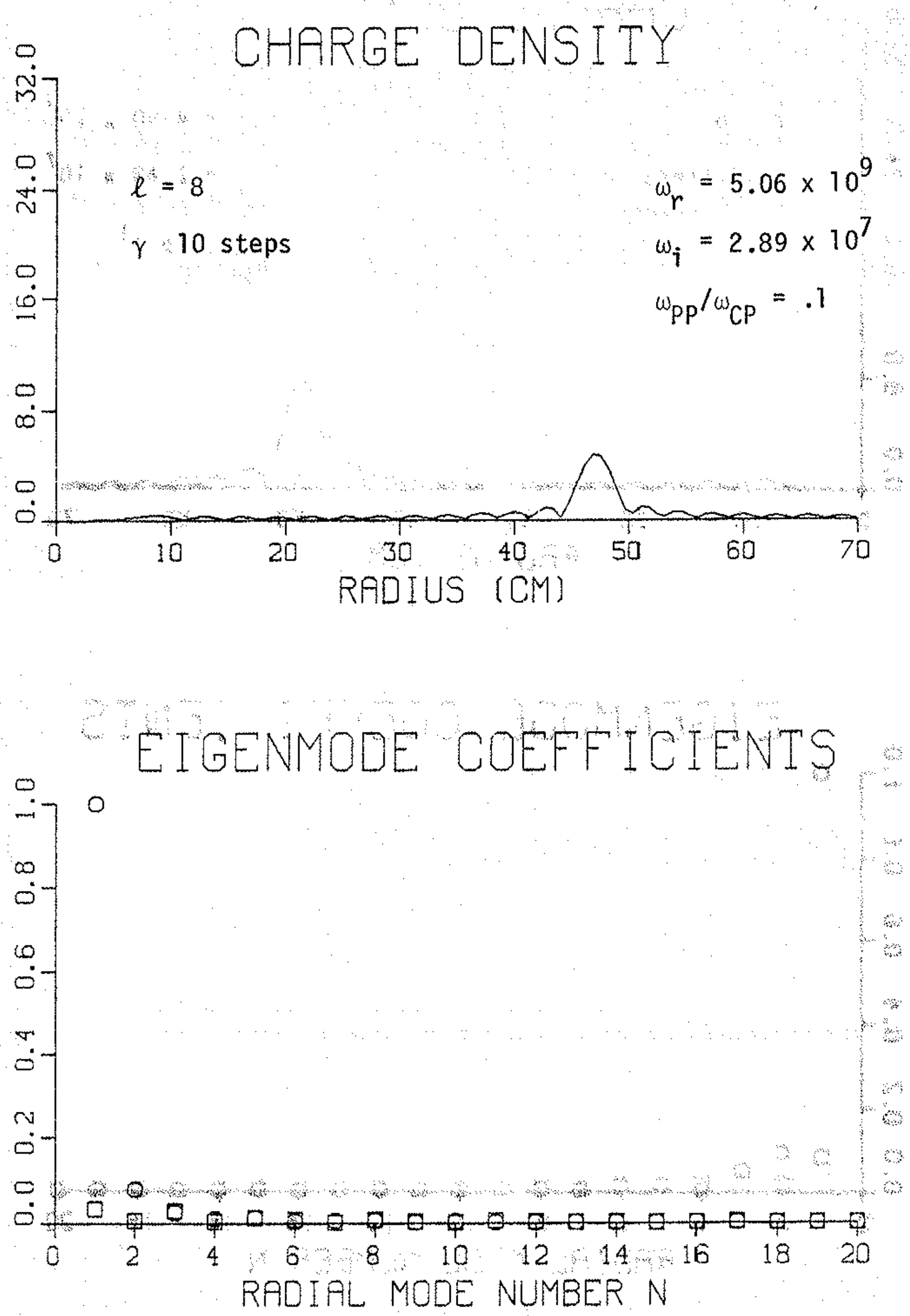
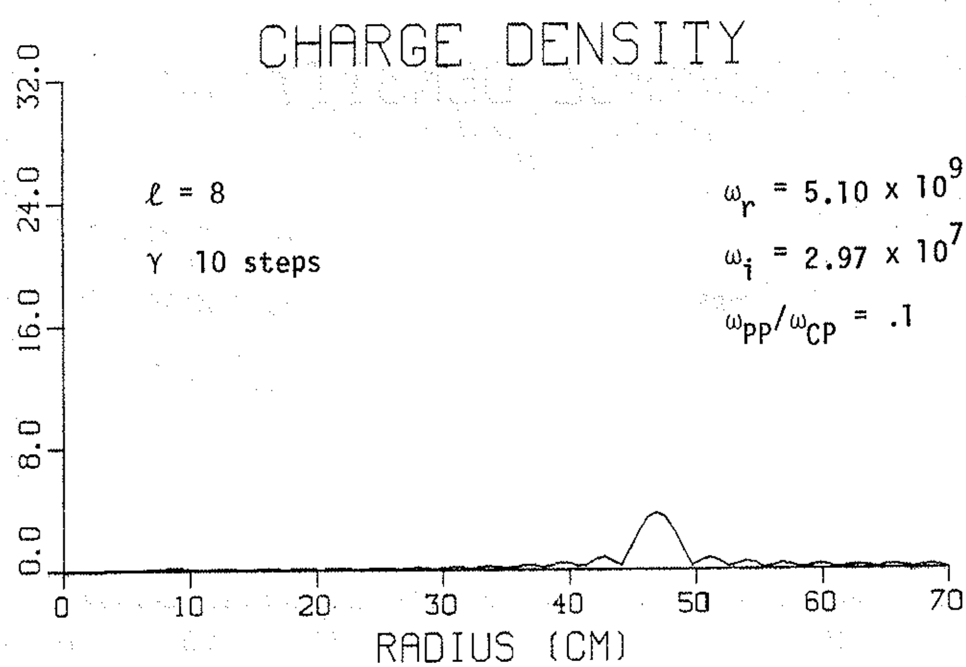

CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 5.06 \times 10^9$
$\omega_i = 2.89 \times 10^7$
$\omega_{PP}/\omega_{CP} = .1$
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.20


CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 5.10 \times 10^9$
$\omega_i = 2.97 \times 10^7$
$\omega_{PP}/\omega_{CP} = .1$
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)

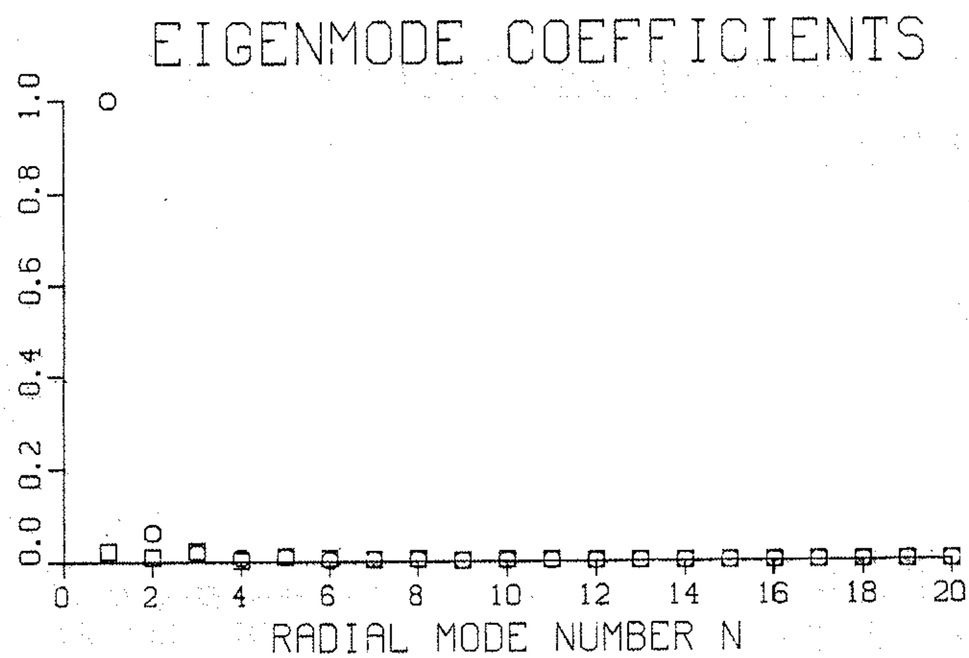
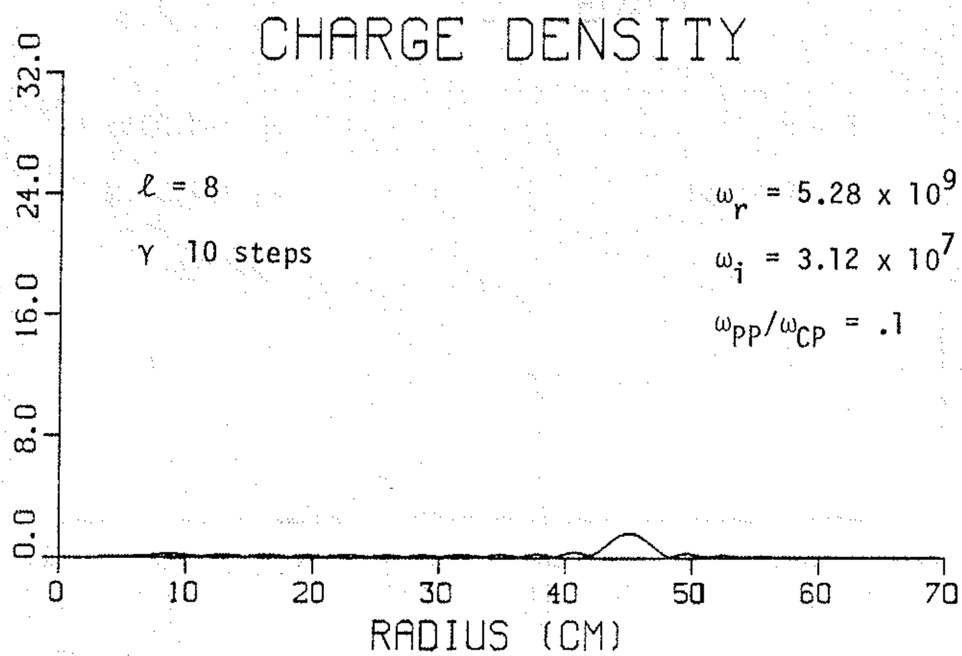

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.21


CHARGE DENSITY
32.0
24.0
16.0
8.0
0.0
ℓ = 8
γ 10 steps
ω_r = 5.28 x 10^9
ω_i = 3.12 x 10^7
ω_{PP}/ω_{CP} = .1
0
10
20
30
40
50
60
70
RADIUS (CM)

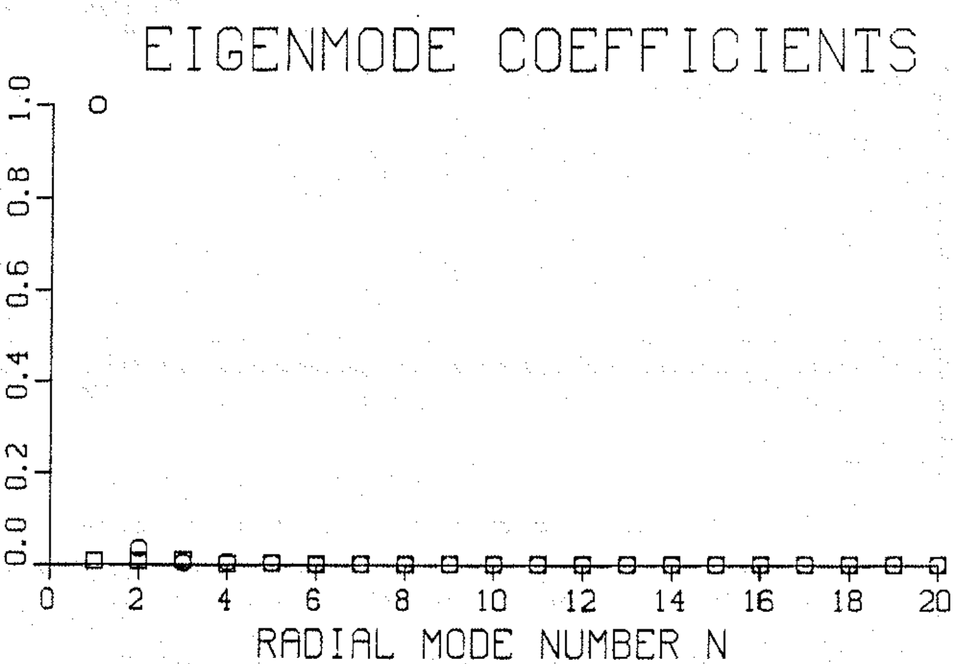
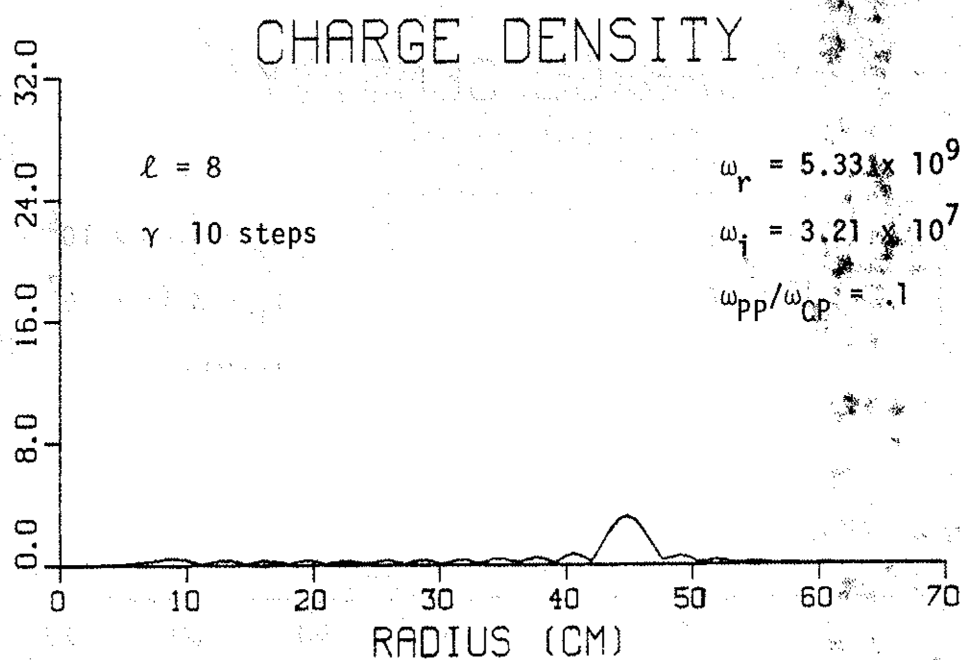

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.22


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 5.33 x 10^9
ω_i = 3.21 x 10^7
ω_{PP}/ω_{CP} = .1
32.0
24.0
16.0
8.0
0.0
0
10
20
30
40
50
60
70
RADIUS (CM)

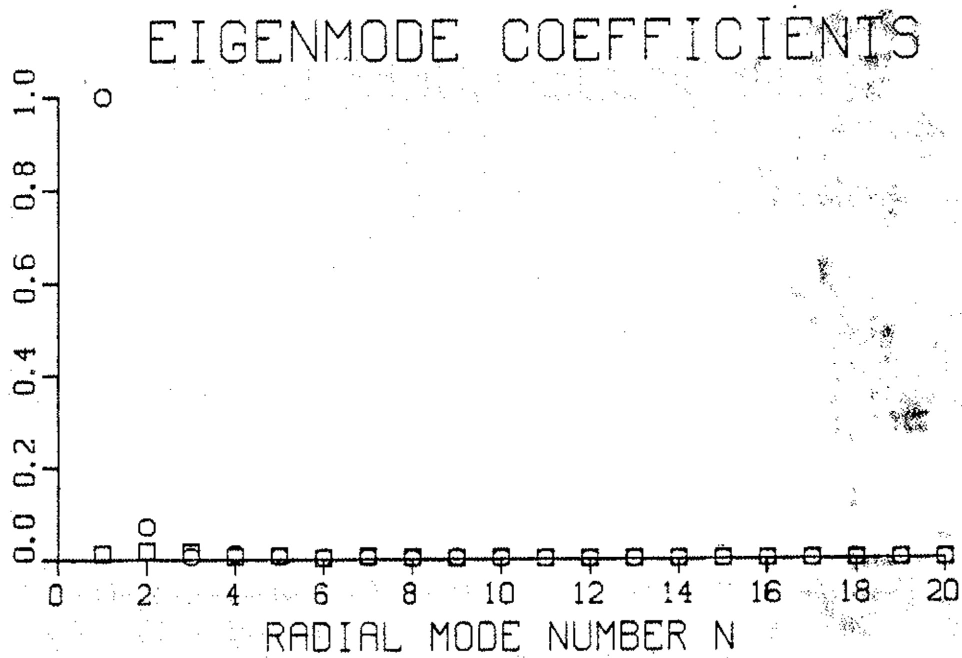
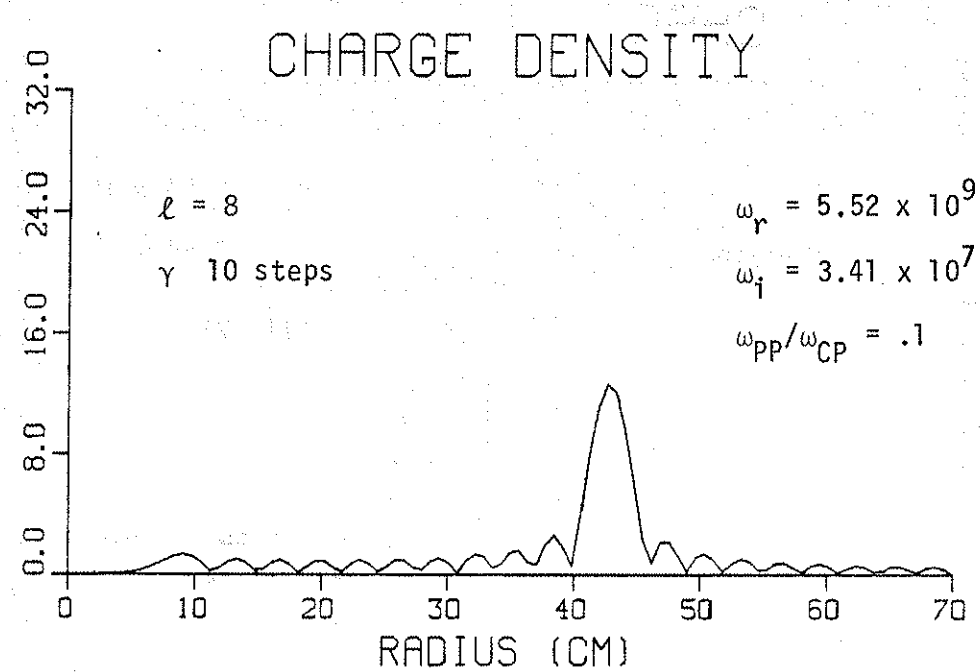

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.23


CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 5.52 \times 10^9$
$\omega_i = 3.41 \times 10^7$
$\omega_{pp}/\omega_{cp} = .1$
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)

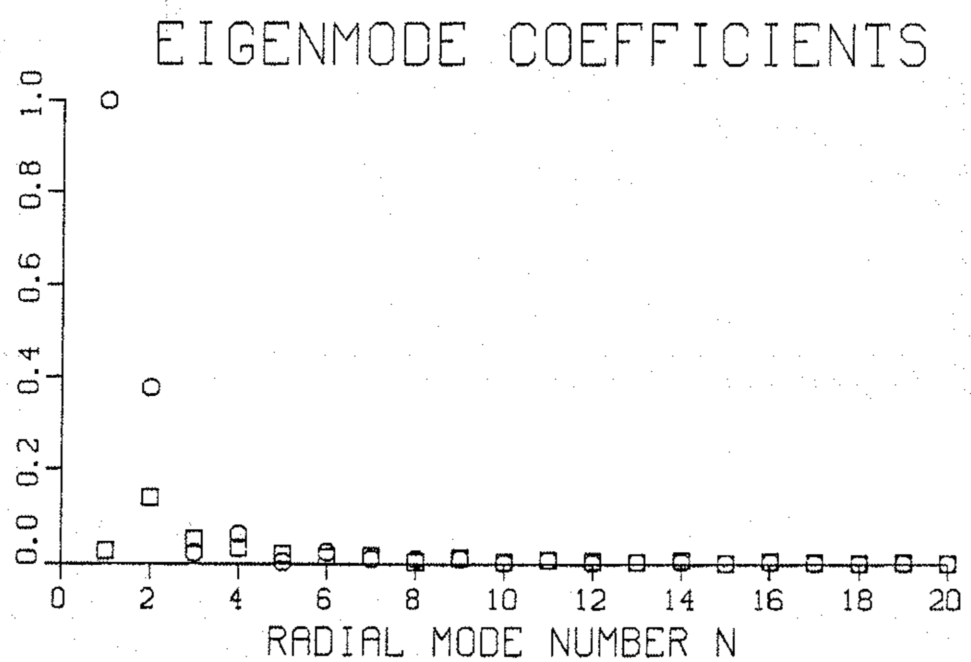

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.24

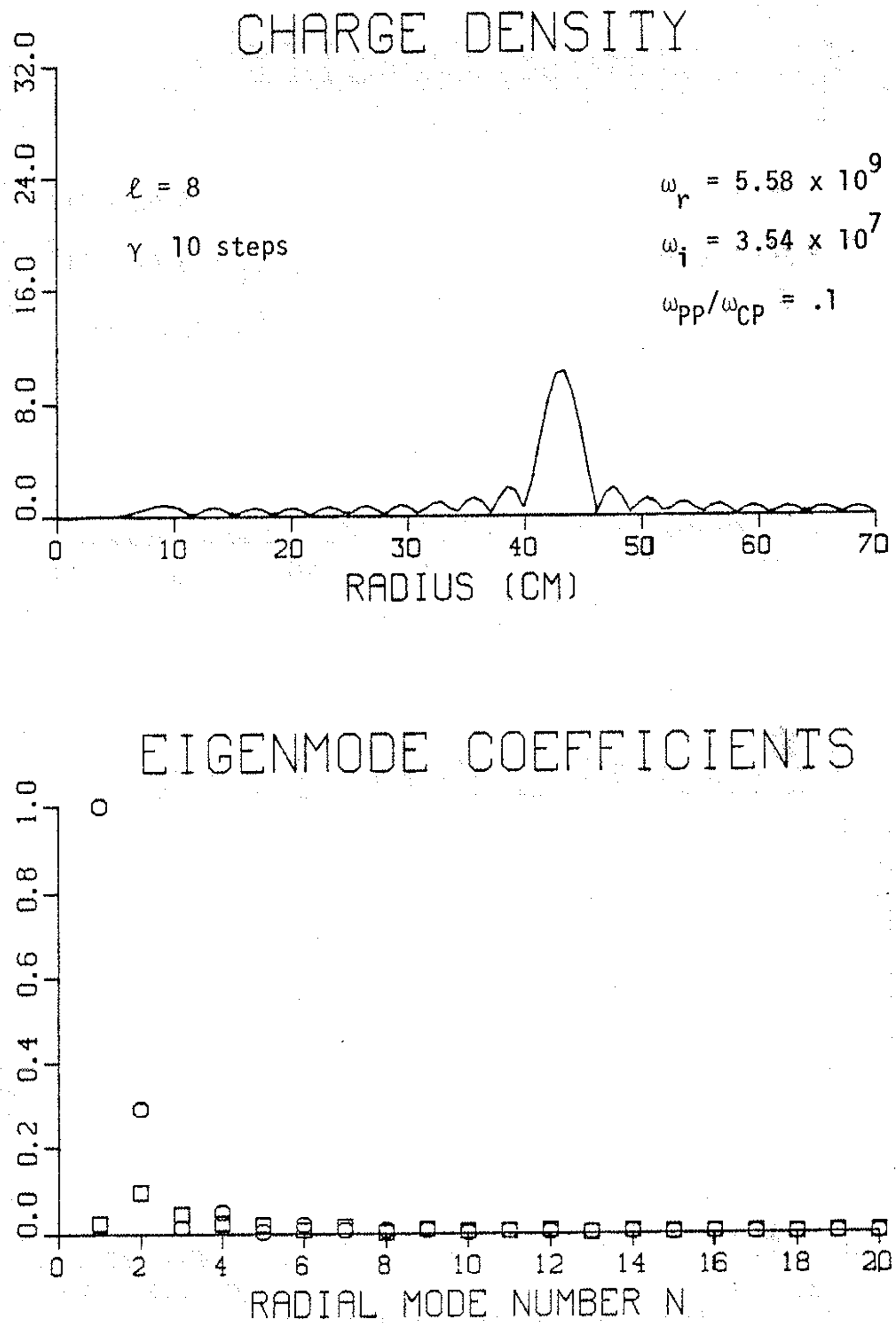
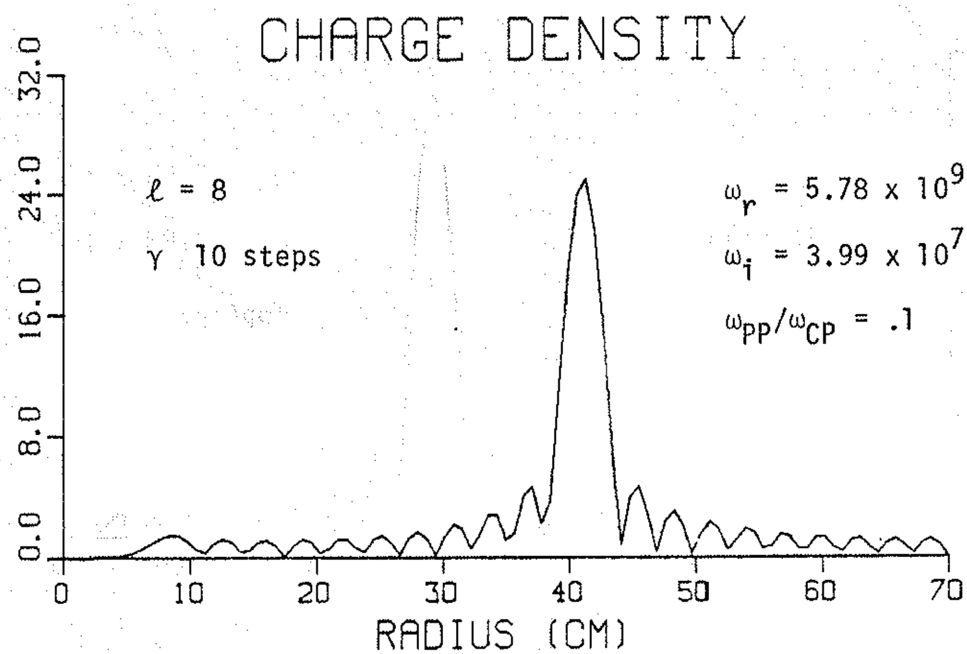

CHARGE DENSITY
32.0
24.0
16.0
8.0
0.0
ℓ = 8
γ 10 steps
ω_r = 5.58 x 10^9
ω_i = 3.54 x 10^7
ω_{PP}/ω_{CP} = .1
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.25


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 5.78 x 10^9
ω_i = 3.99 x 10^7
ω_{PP}/ω_{CP} = .1
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)

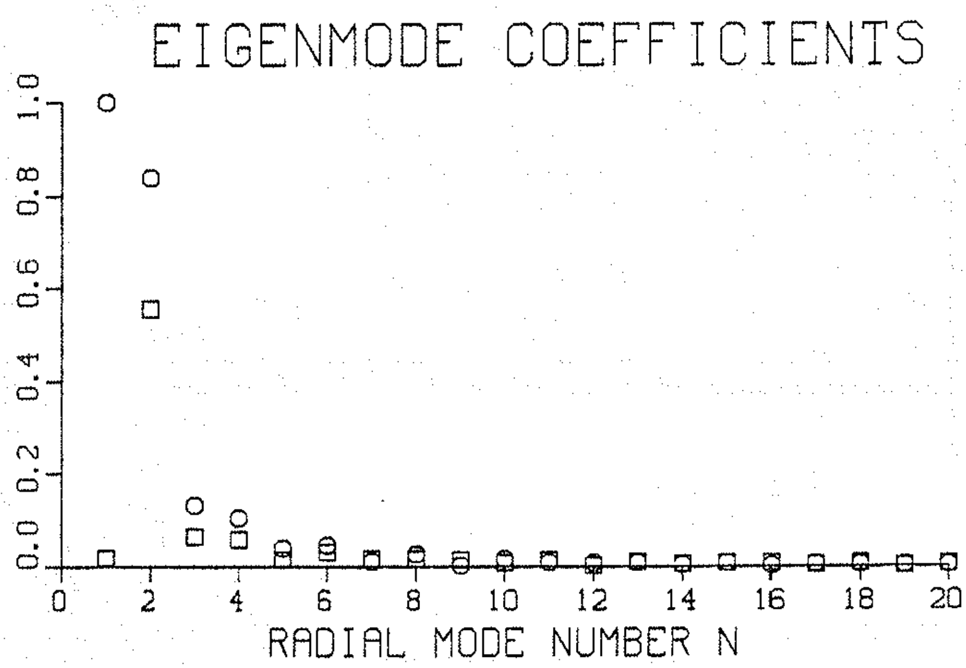
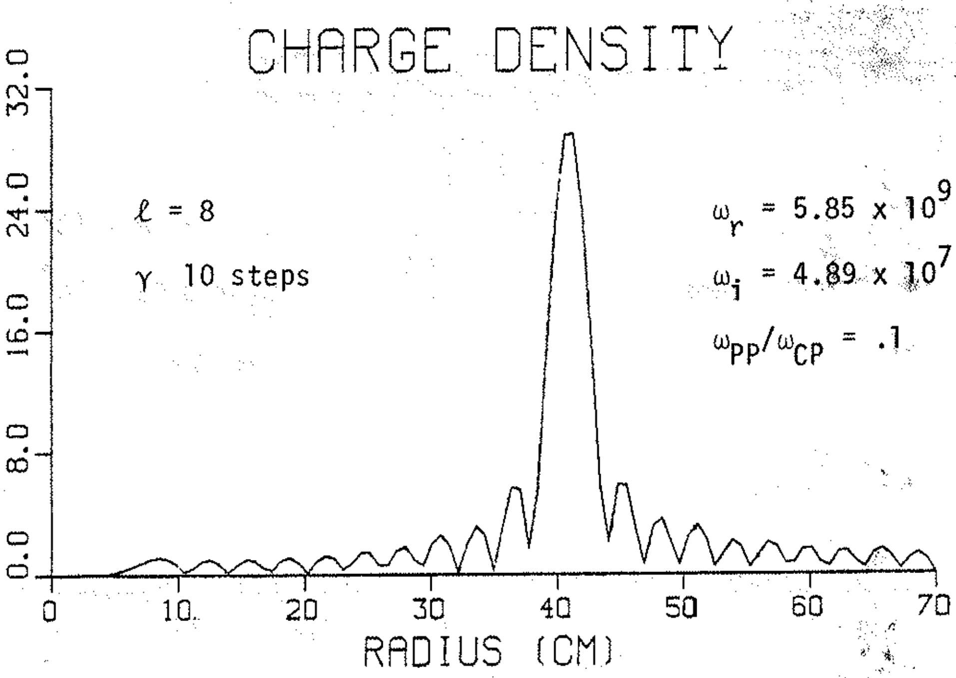

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.26


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 5.85 x 10^9
ω_i = 4.89 x 10^7
ω_{pp}/ω_{cp} = .1
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)

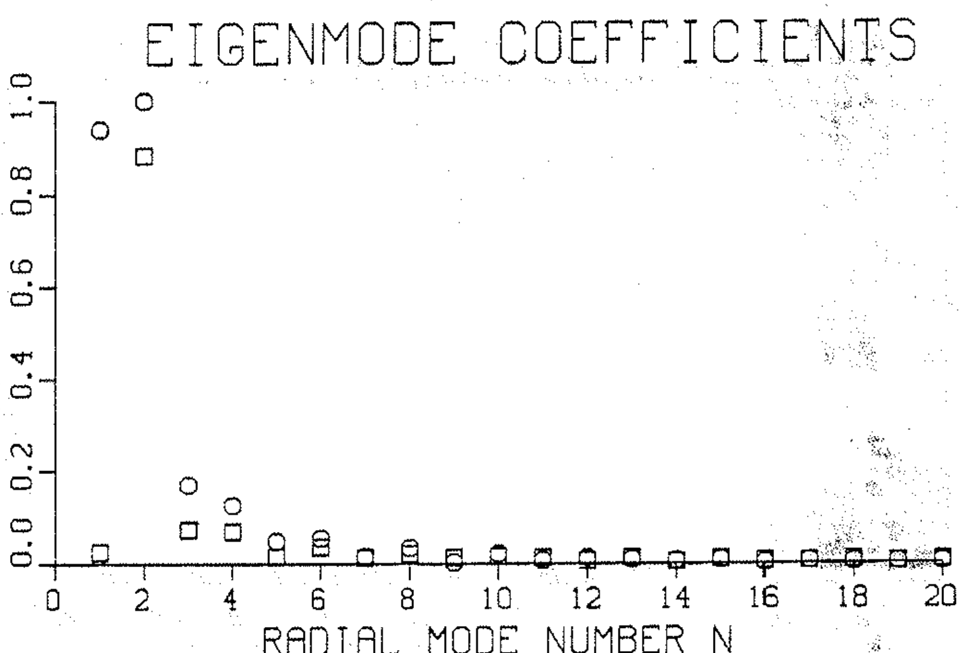

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.27

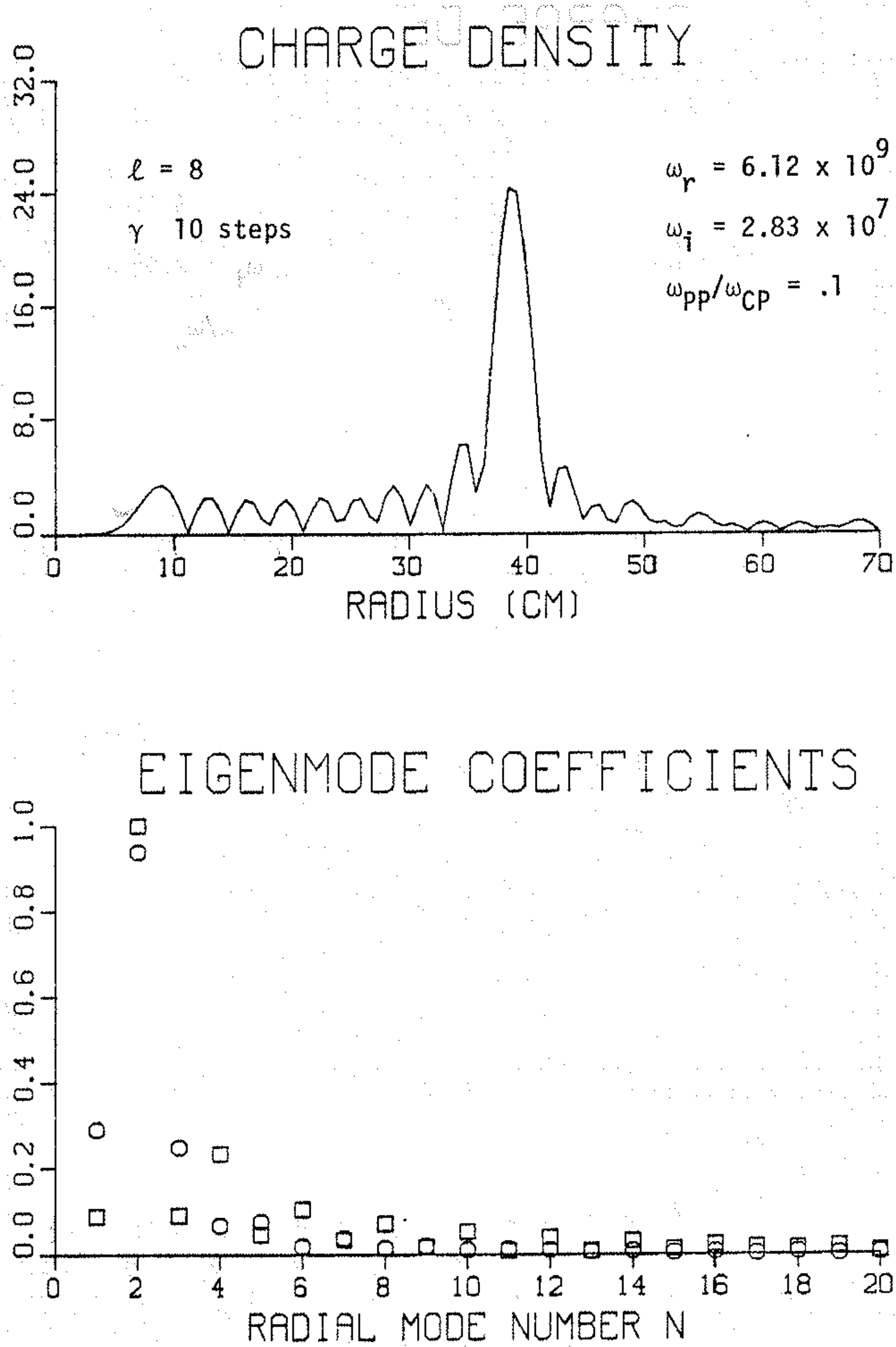
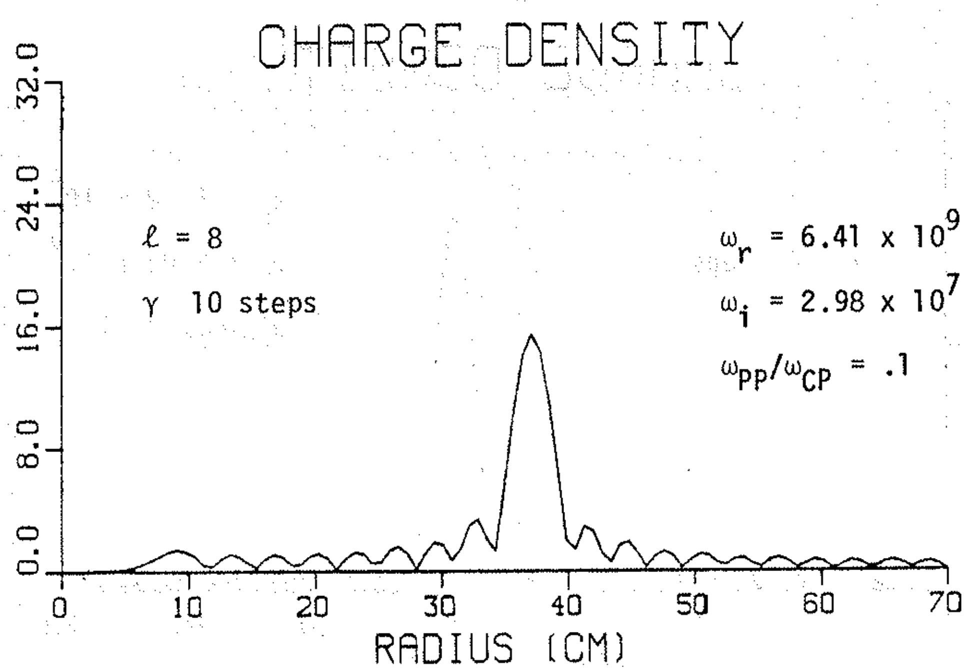

CHARGE DENSITY
32.0
24.0
16.0
8.0
0.0
ℓ = 8
γ 10 steps
ω_r = 6.12 x 10^9
ω_i = 2.83 x 10^7
ω_{pp}/ω_{CP} = .1
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.28


CHARGE DENSITY
32.0
24.0
16.0
8.0
0.0
ℓ = 8
γ 10 steps
$\omega_r = 6.41 \times 10^9$
$\omega_i = 2.98 \times 10^7$
$\omega_{PP}/\omega_{CP} = .1$
0
10
20
30
40
50
60
70
RADIUS (CM)

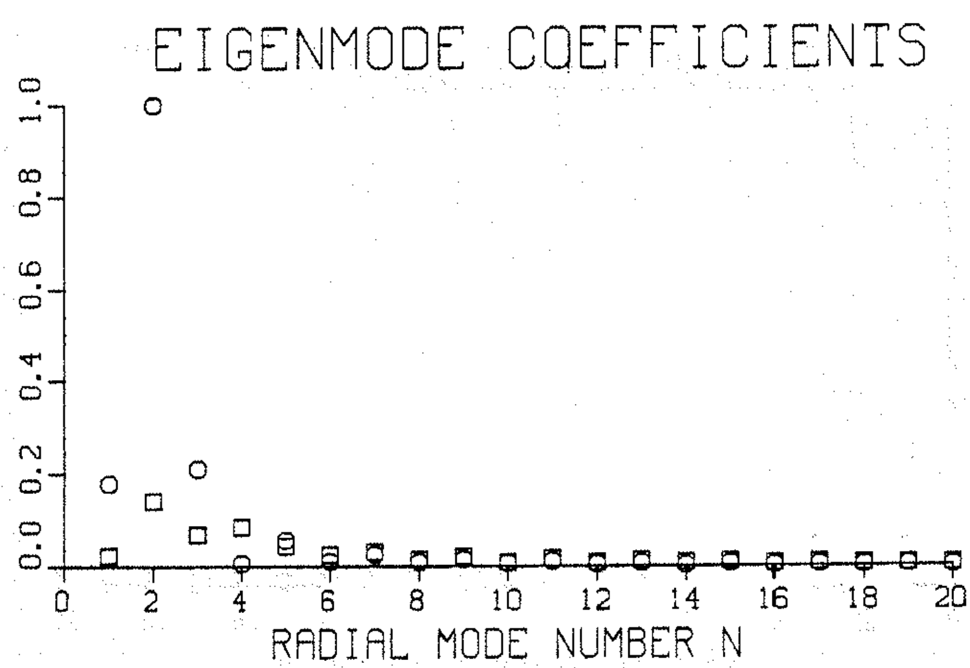
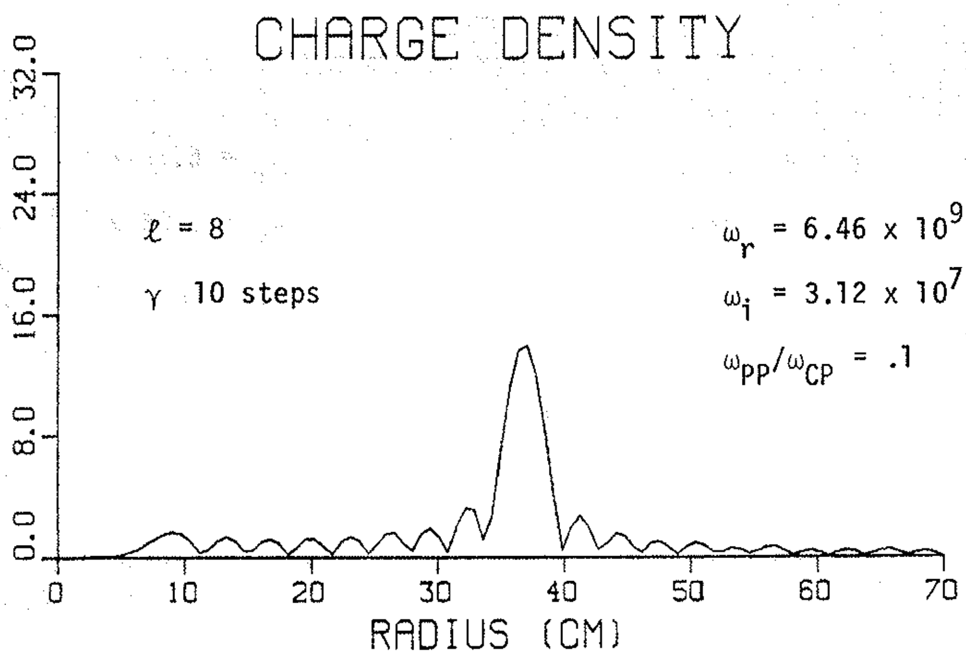

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.29


CHARGE DENSITY
0.0
8.0
16.0
24.0
32.0
ℓ = 8
γ 10 steps
ω_r = 6.46 x 10^9
ω_i = 3.12 x 10^7
ω_PP/ω_CP = .1
0
10
20
30
40
50
60
70
RADIUS (CM)

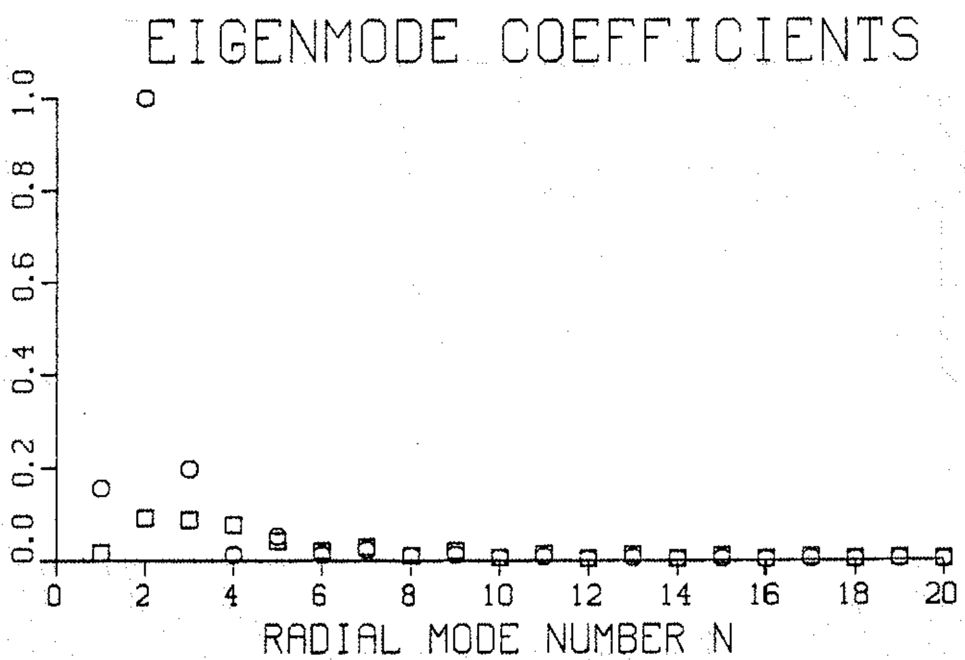
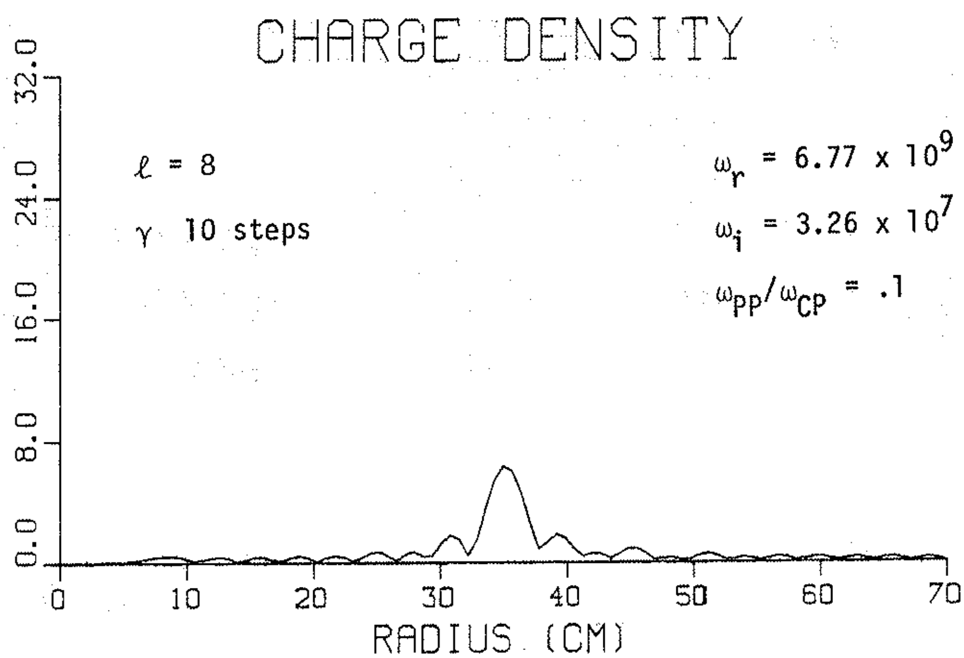

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.30


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 6.77 x 10^9
ω_i = 3.26 x 10^7
ω_{PP}/ω_{CP} = .1
32.0
24.0
16.0
8.0
0.0
0
10
20
30
40
50
60
70
RADIUS (CM)

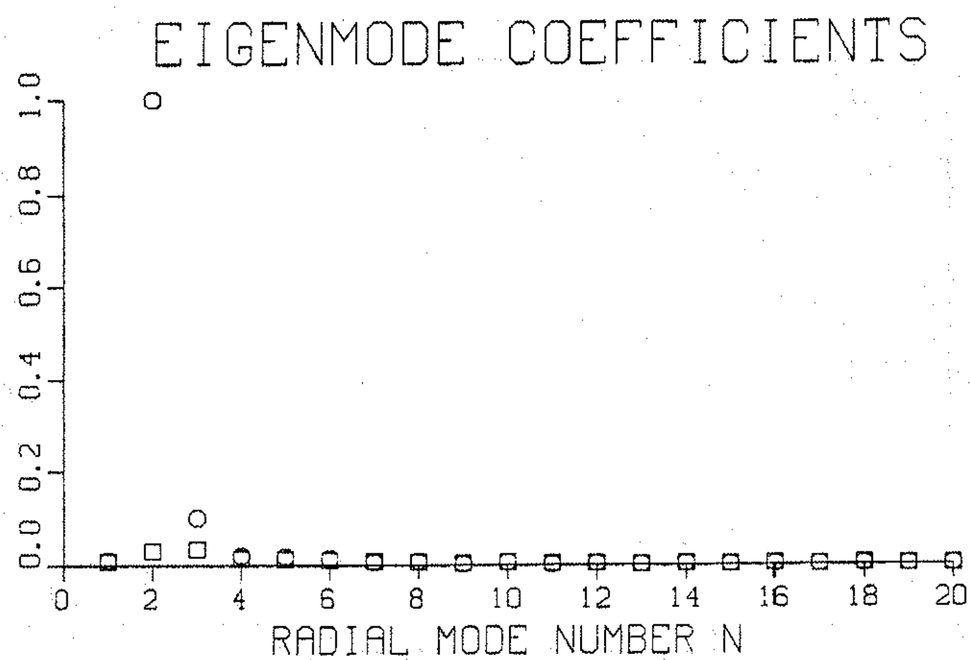

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.31

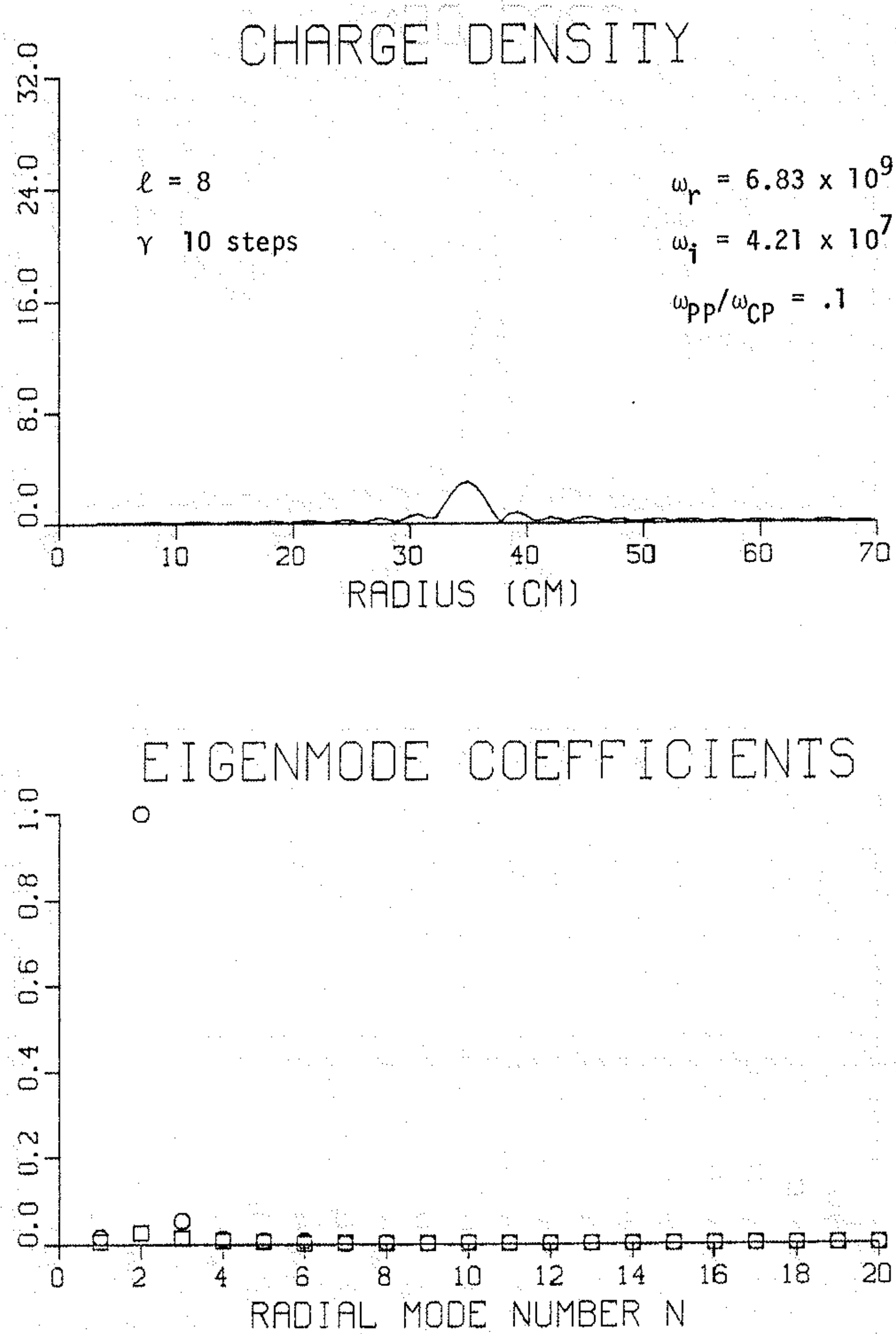
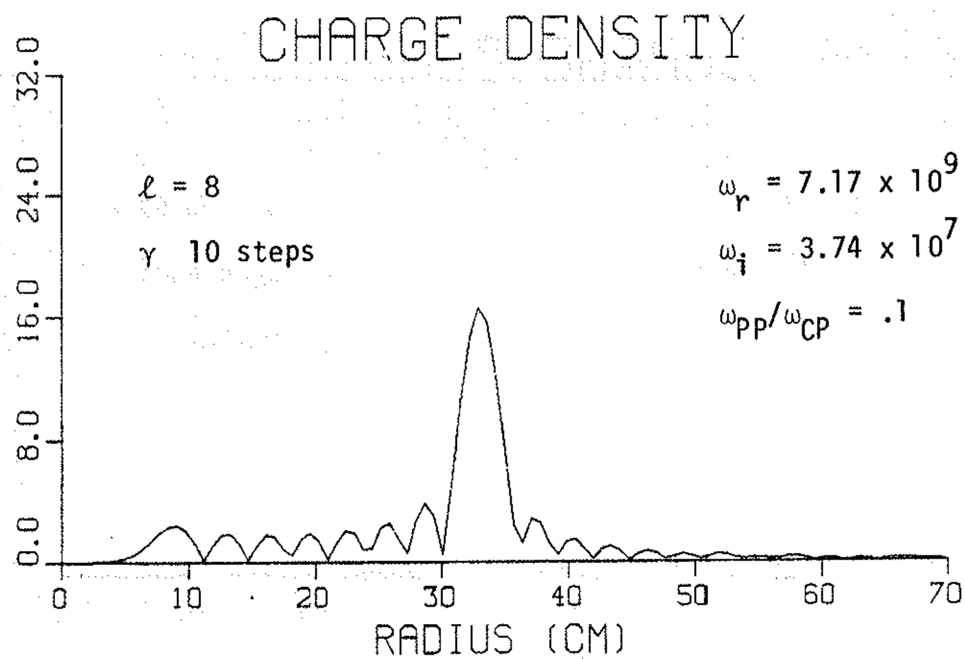

CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 6.83 x 10^9
ω_i = 4.21 x 10^7
ω_{PP}/ω_{CP} = .1
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.32


CHARGE DENSITY
$\ell = 8$
γ 10 steps
$\omega_r = 7.17 \times 10^9$
$\omega_i = 3.74 \times 10^7$
$\omega_{PP}/\omega_{CP} = .1$
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)

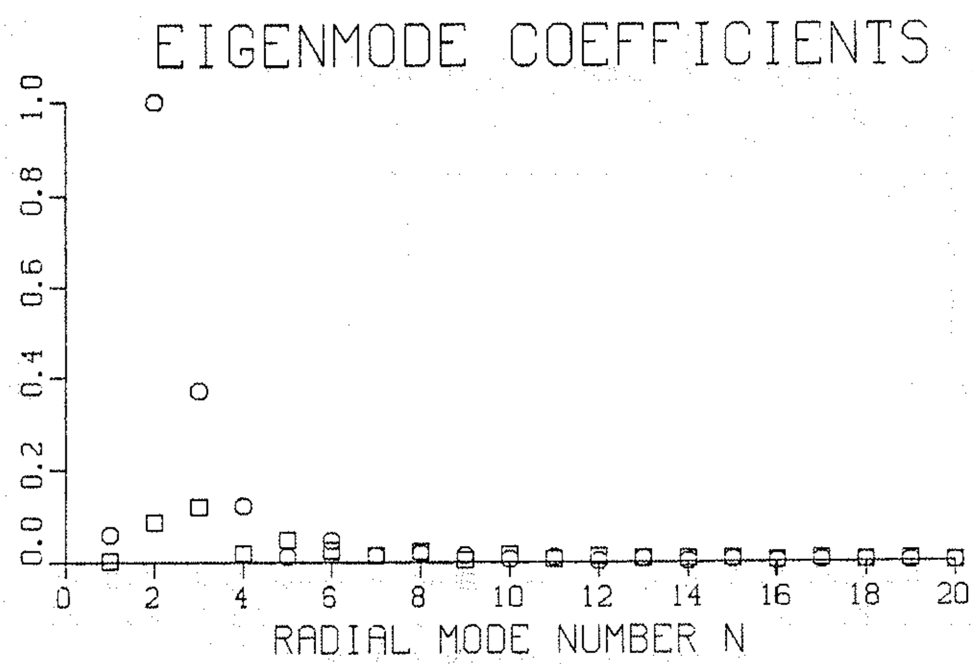

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.33

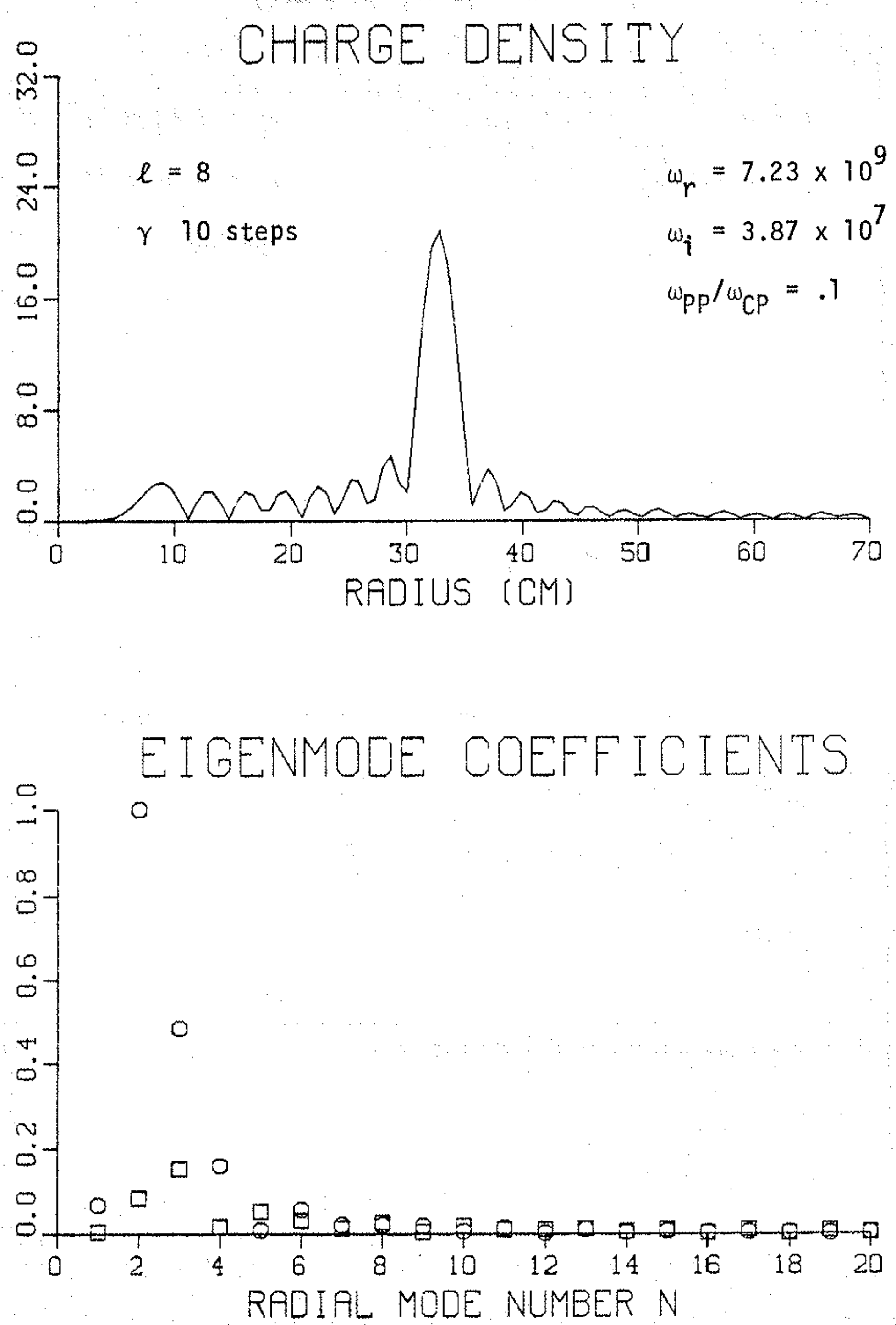
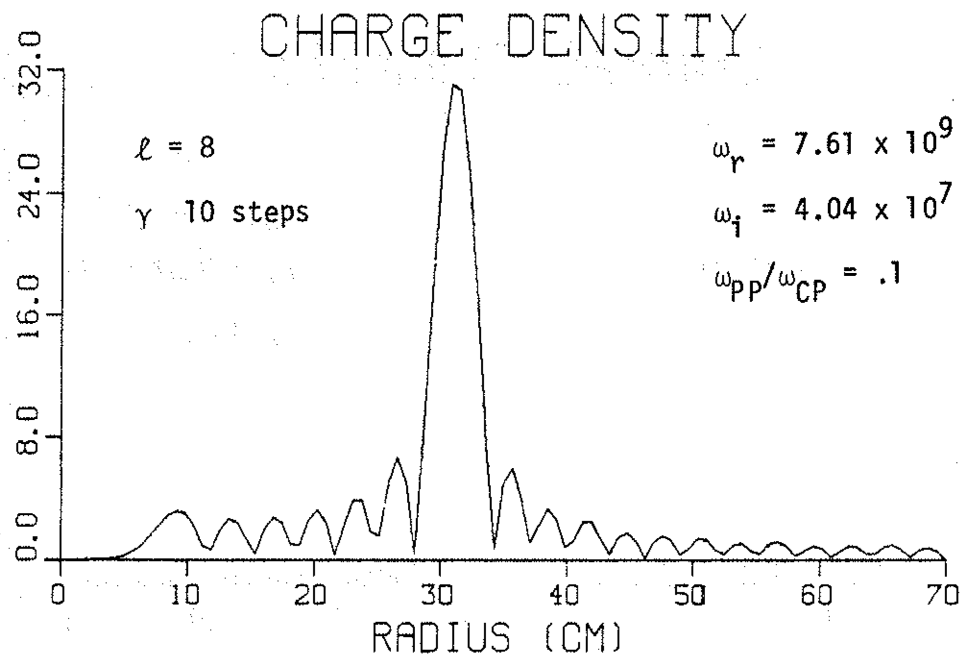

CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 7.23 x 10^9
ω_i = 3.87 x 10^7
ω_PP/ω_CP = .1
RADIUS (CM)
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
EIGENMODE COEFFICIENTS
RADIAL MODE NUMBER N
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20

Fig. 3.34


CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 7.61 \times 10^9$
$\omega_i = 4.04 \times 10^7$
$\omega_{pp}/\omega_{cp} = .1$
0.0
8.0
16.0
24.0
32.0
0
10
20
30
40
50
60
70
RADIUS (CM)

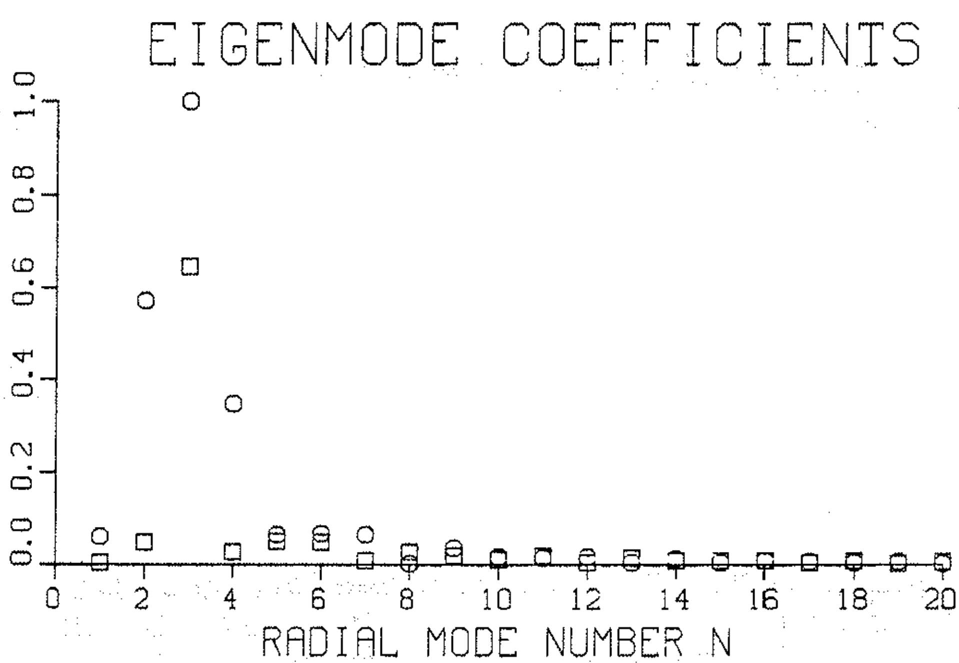

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.35

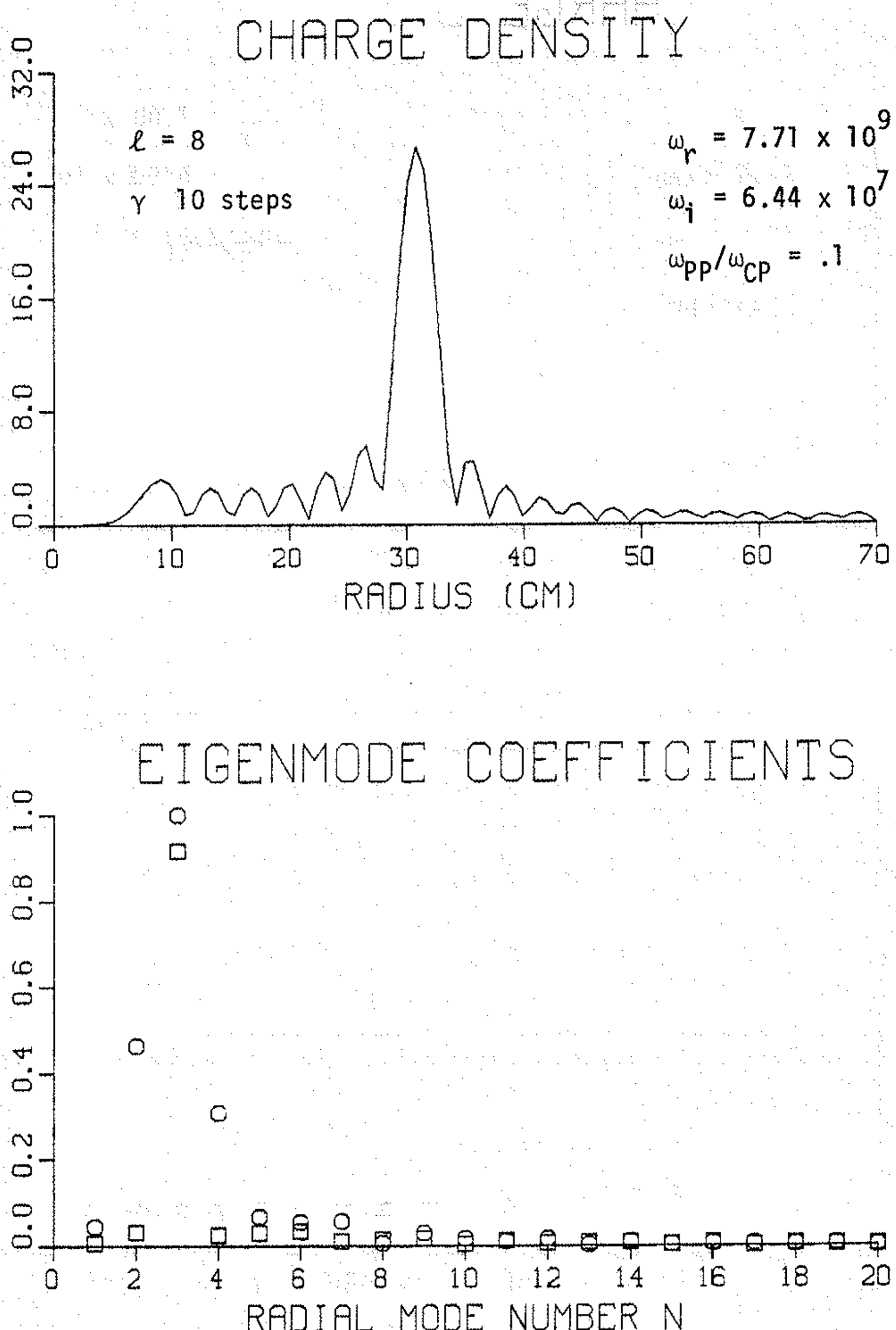
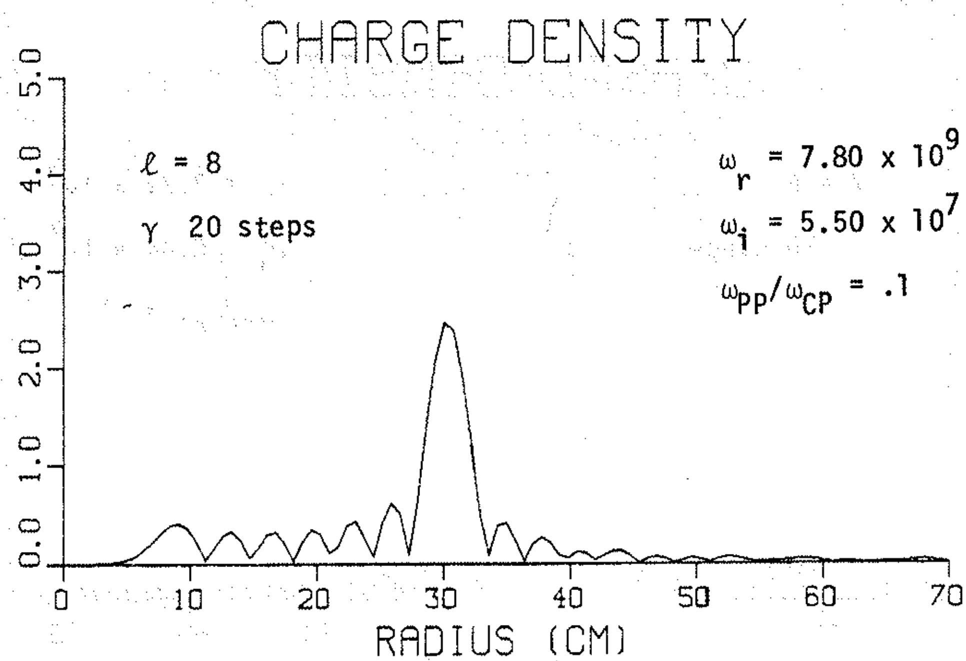

CHARGE DENSITY
32.0
24.0
16.0
8.0
0.0
ℓ = 8
γ 10 steps
ω_r = 7.71 x 10^9
ω_i = 6.44 x 10^7
ω_PP/ω_CP = .1
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.36


CHARGE DENSITY
ℓ = 8
γ 20 steps
ω_r = 7.80 x 10^9
ω_i = 5.50 x 10^7
ω_{PP}/ω_{CP} = .1
0.0
1.0
2.0
3.0
4.0
5.0
0
10
20
30
40
50
60
70
RADIUS (CM)

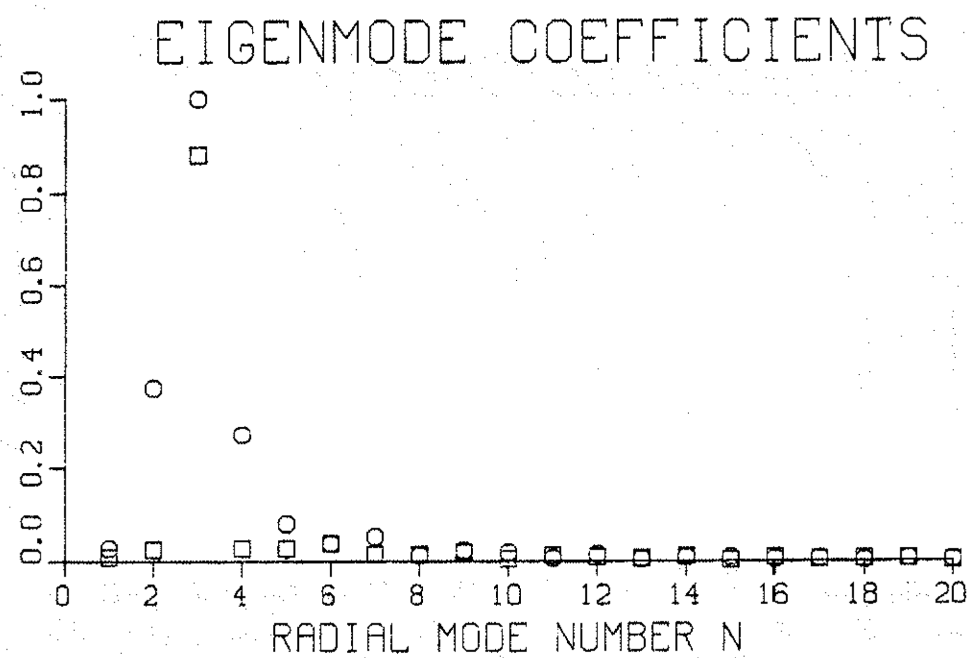
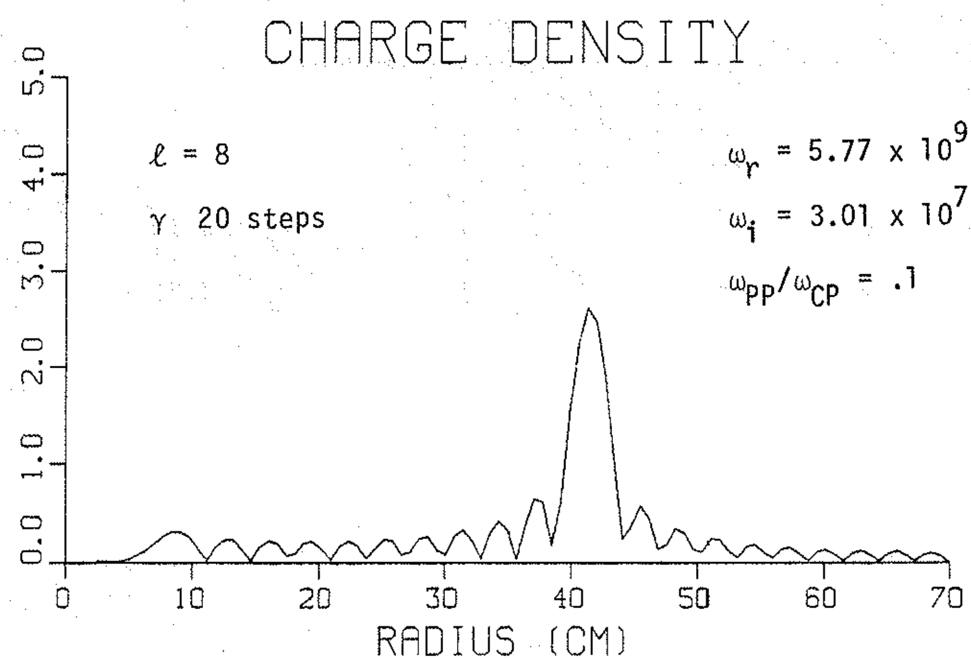

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.37


CHARGE DENSITY
ℓ = 8
γ 20 steps
ωr = 5.77 x 10^9
ωi = 3.01 x 10^7
ωPP/ωCP = .1
0.0
1.0
2.0
3.0
4.0
5.0
0
10
20
30
40
50
60
70
RADIUS (CM)

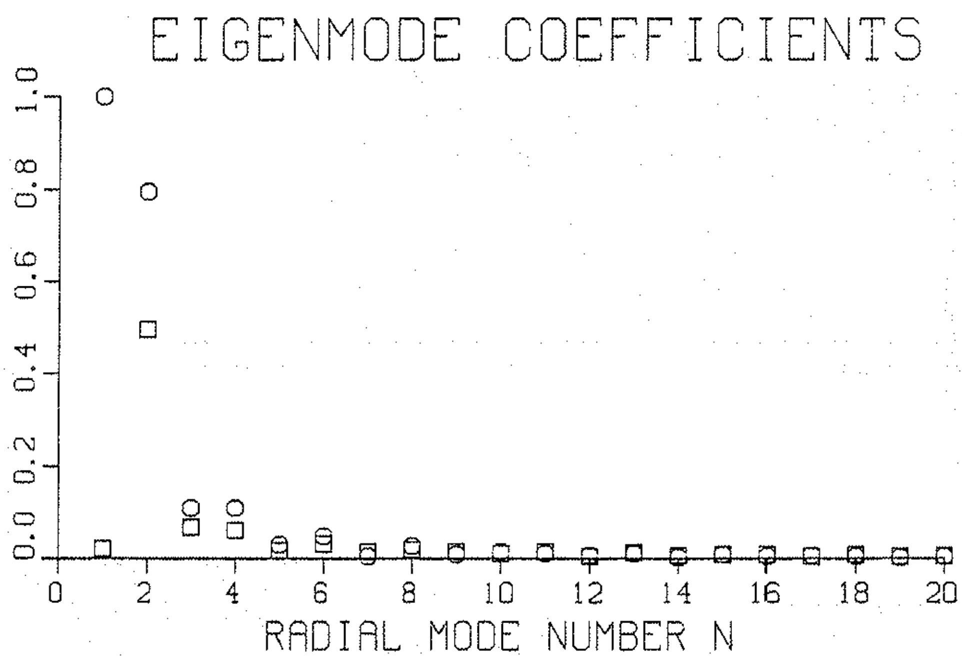
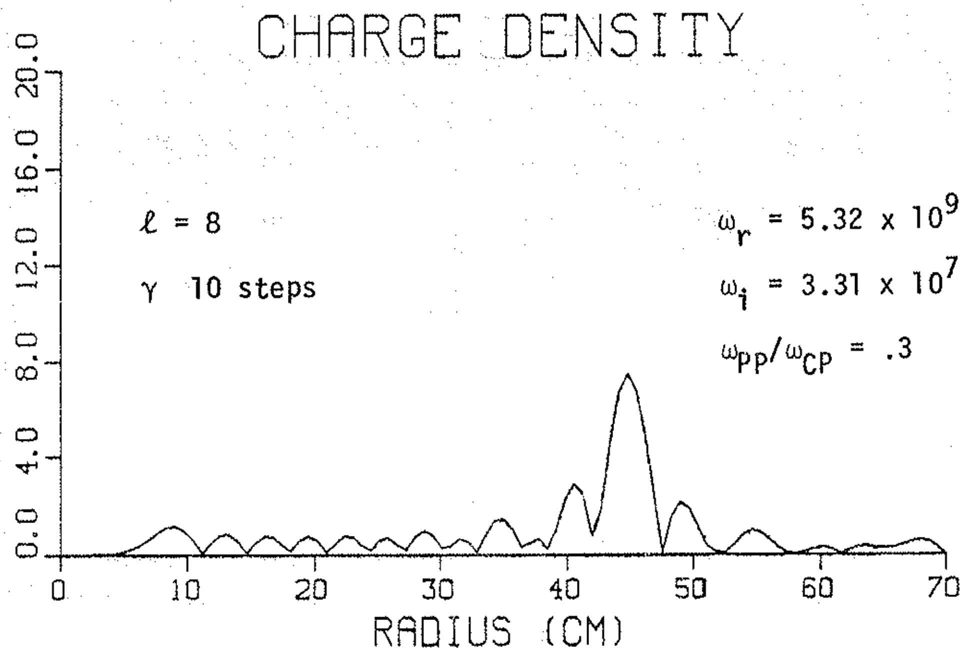

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.38


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 5.32 x 10^9
ω_i = 3.31 x 10^7
ω_pp/ω_CP = .3
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)

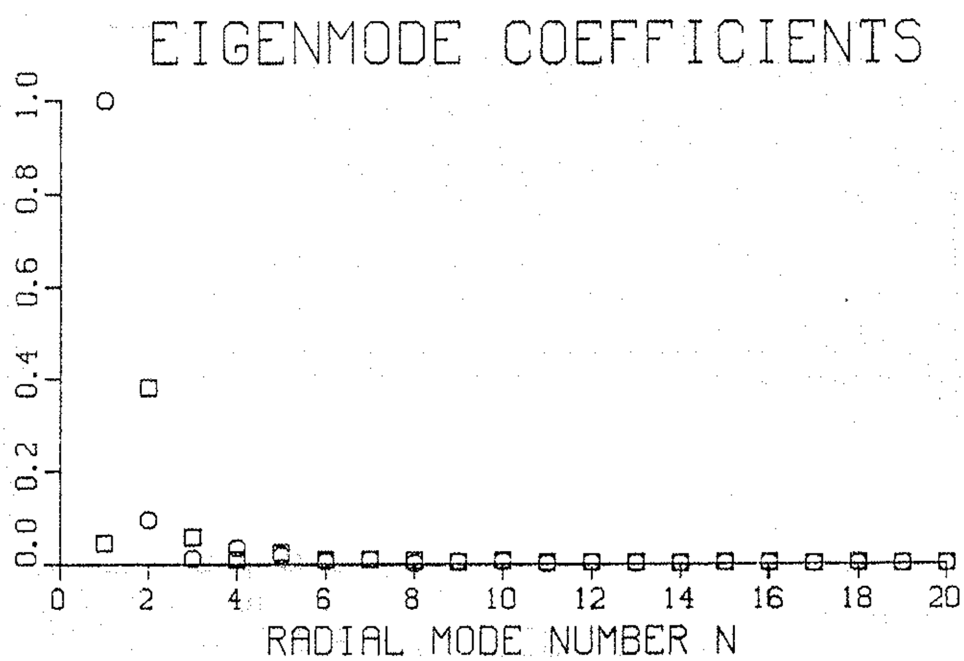

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.39

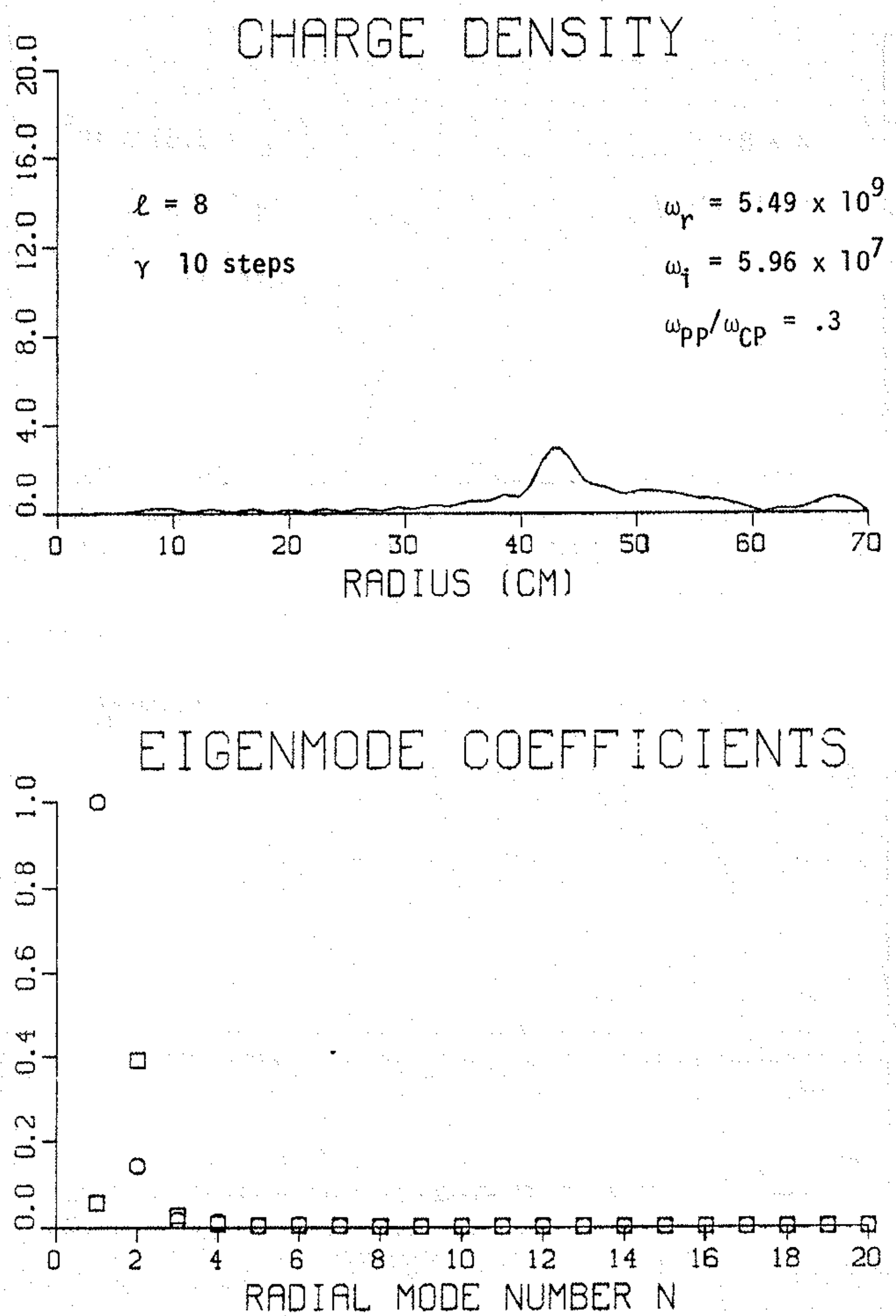
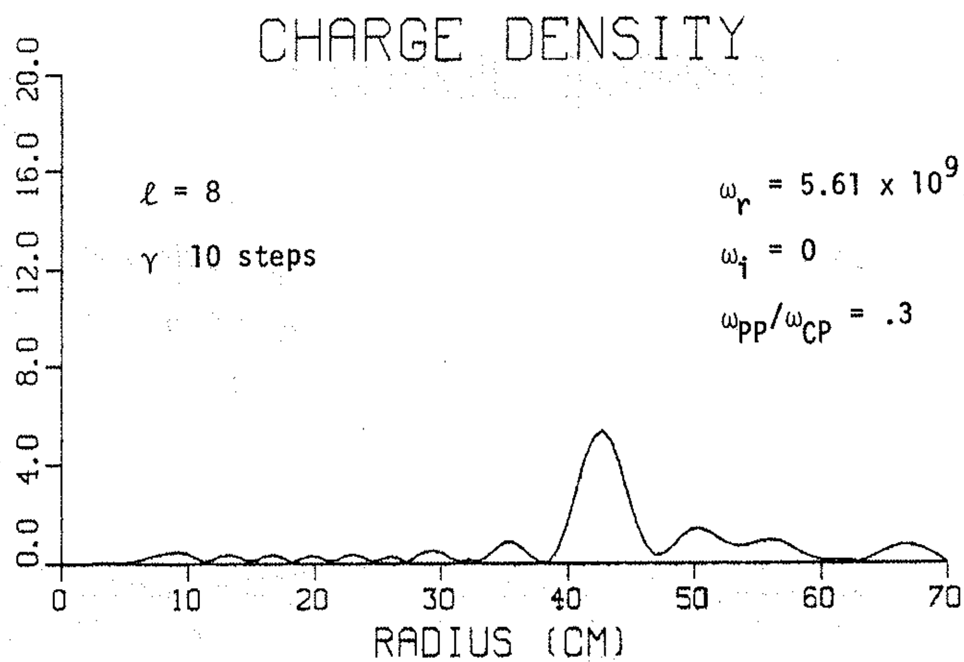

CHARGE DENSITY
20.0
16.0
12.0
8.0
4.0
0.0
ℓ = 8
γ 10 steps
ω_r = 5.49 x 10^9
ω_i = 5.96 x 10^7
ω_{PP}/ω_{CP} = .3
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.40


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 5.61 x 10^9
ω_i = 0
ω_{PP}/ω_{CP} = .3
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)

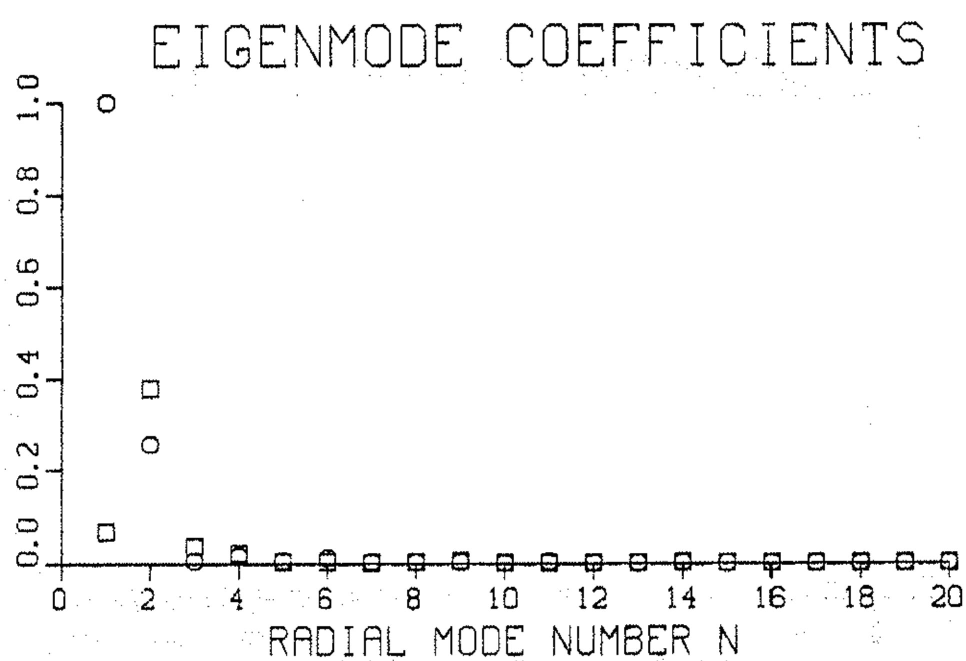
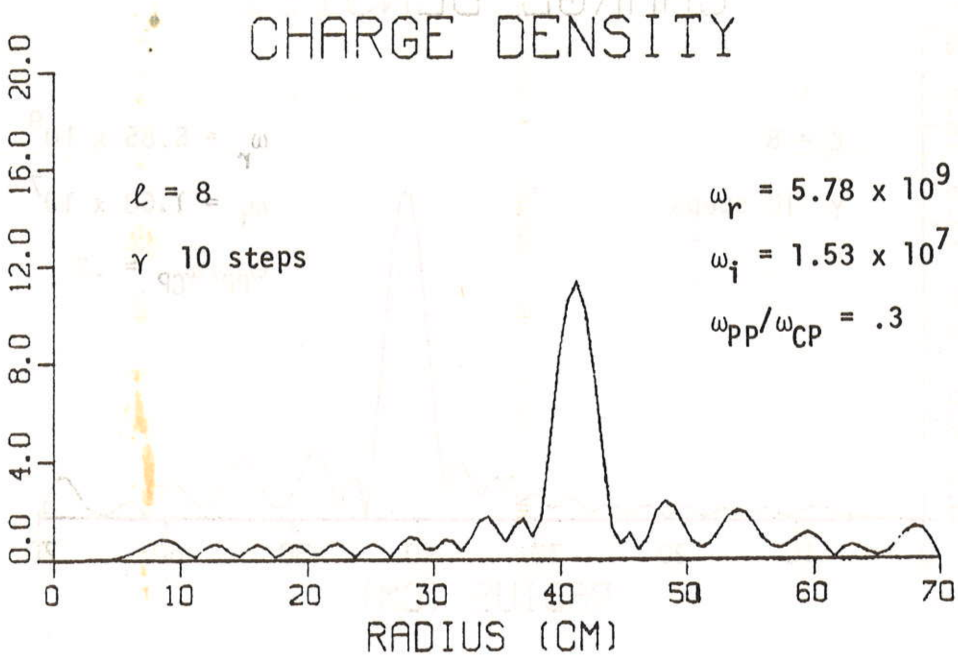

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.41


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 5.78 x 10^9
ω_i = 1.53 x 10^7
ω_PP/ω_CP = .3
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)

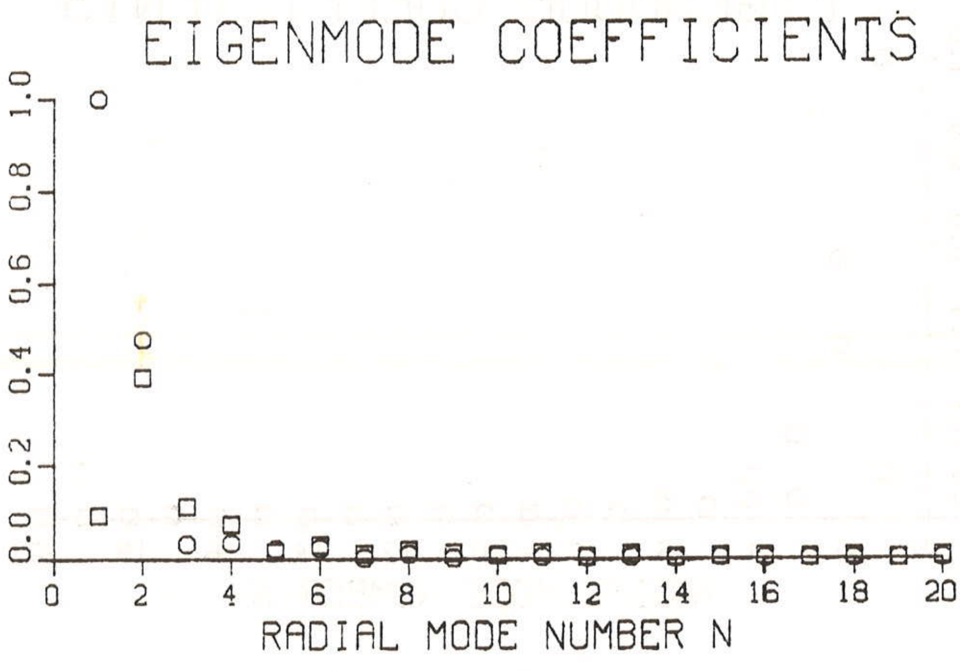
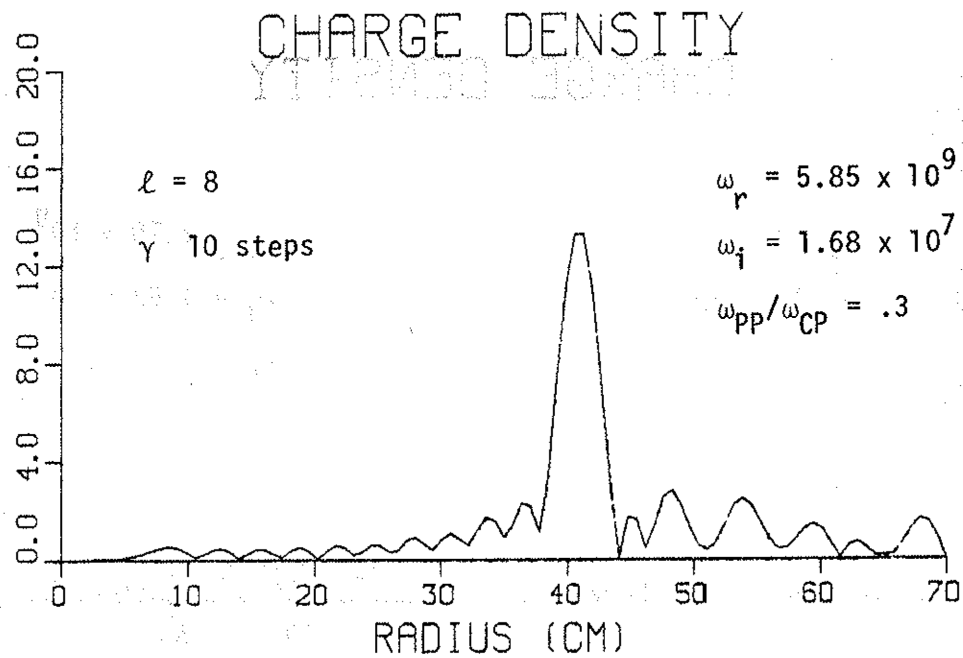

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.42


CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 5.85 \times 10^9$
$\omega_i = 1.68 \times 10^7$
$\omega_{PP}/\omega_{CP} = .3$
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)

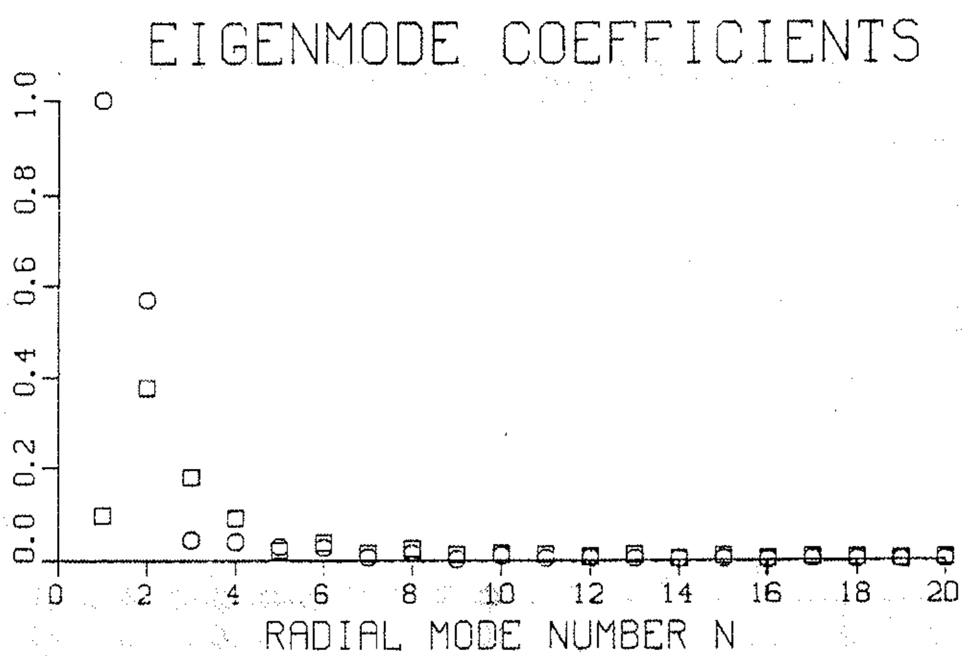
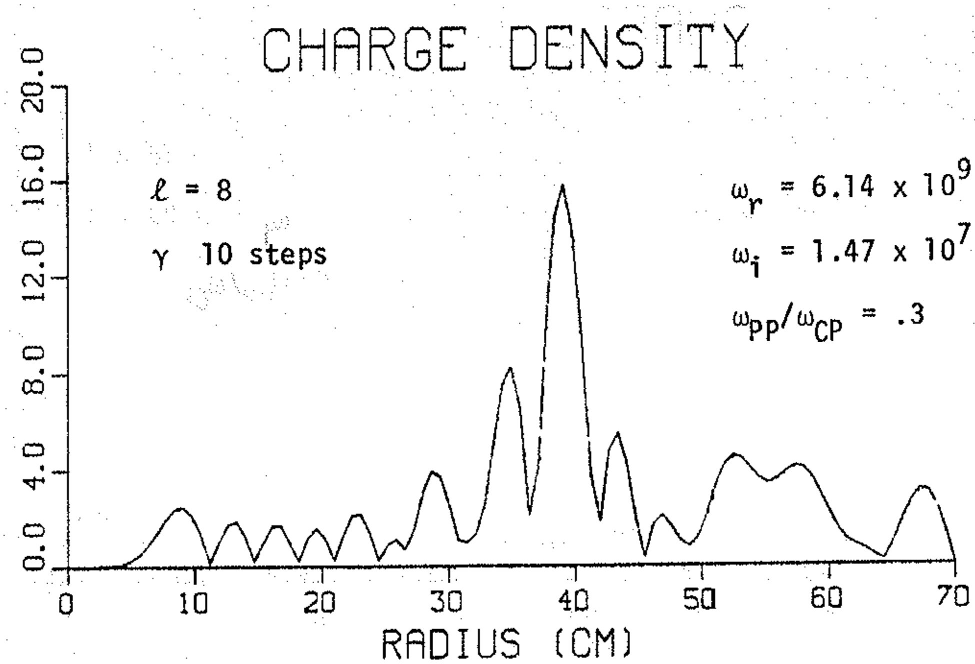

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.43


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 6.14 x 10^9
ω_i = 1.47 x 10^7
ω_pp/ω_CP = .3
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)

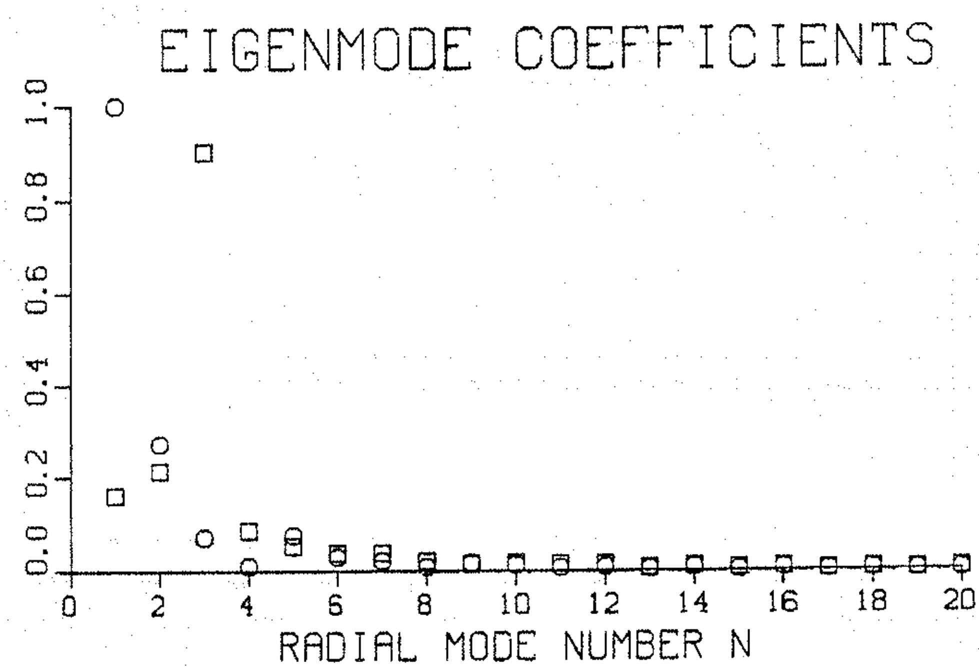

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.44

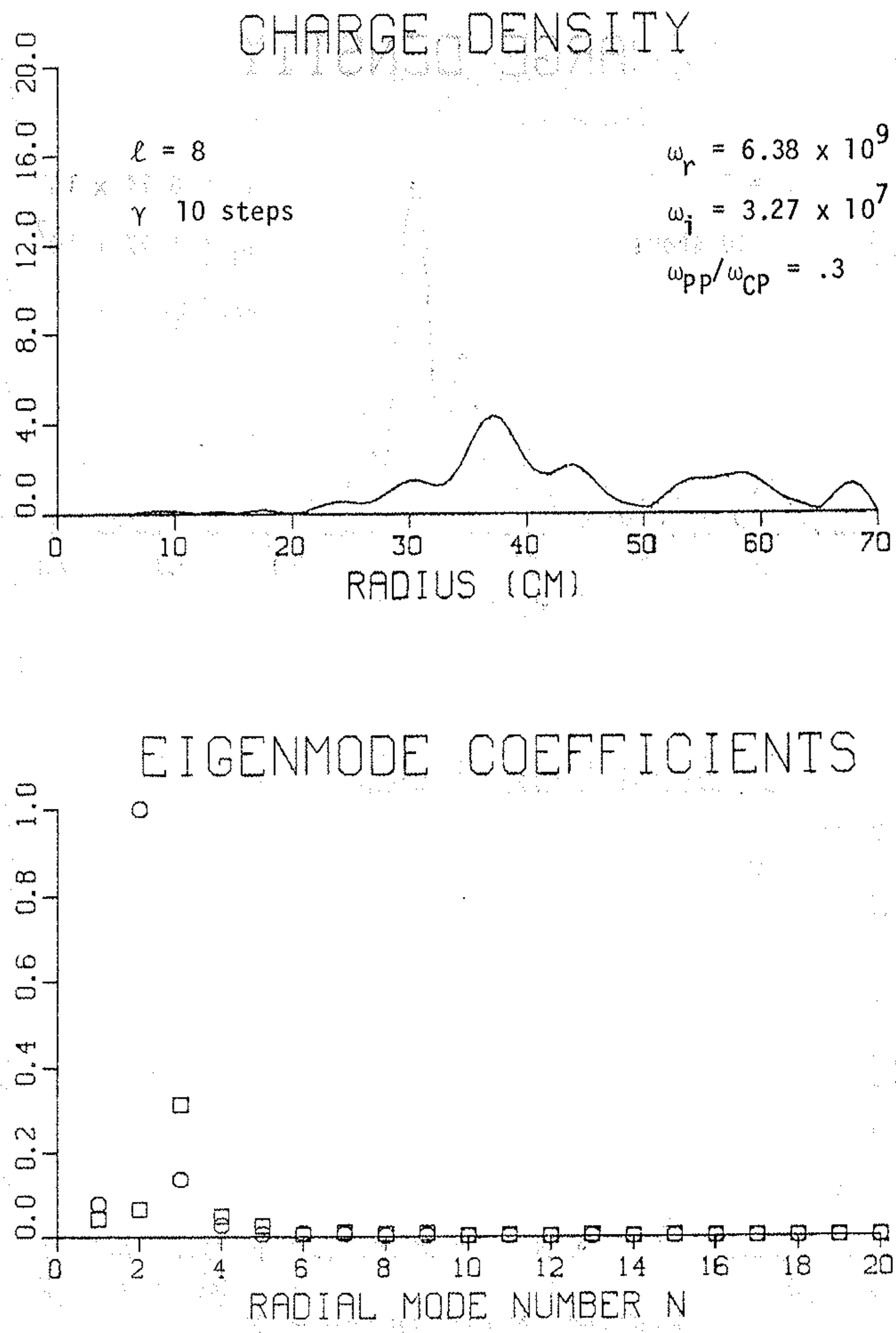
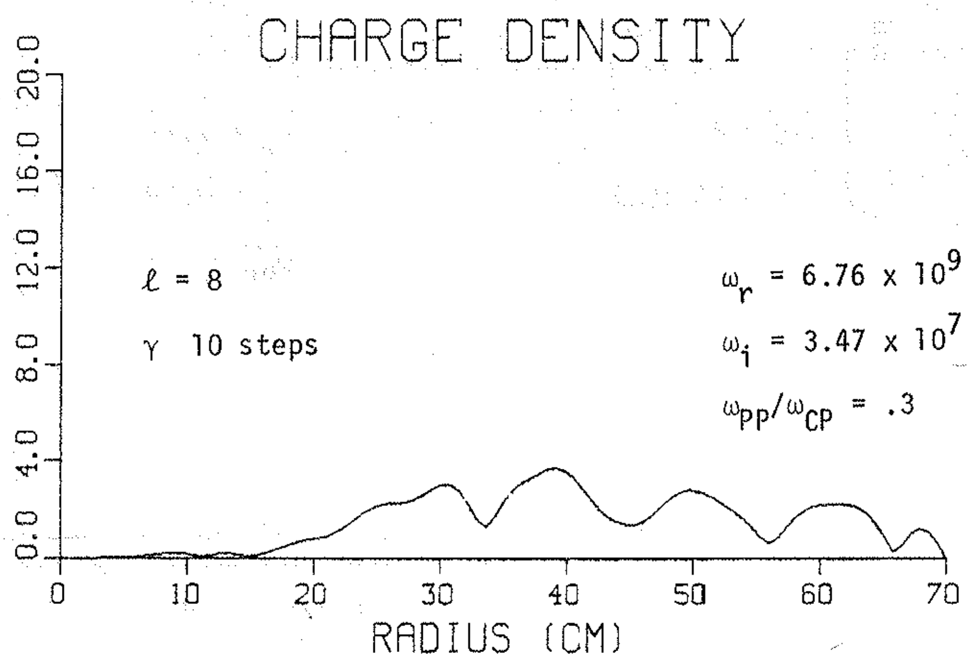

CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 6.38 x 10^9
ω_i = 3.27 x 10^7
ω_pp/ω_CP = .3
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.45


CHARGE DENSITY
20.0
16.0
12.0
8.0
4.0
0.0
ℓ = 8
γ 10 steps
ω_r = 6.76 x 10^9
ω_i = 3.47 x 10^7
ω_pp/ω_CP = .3
0
10
20
30
40
50
60
70
RADIUS (CM)

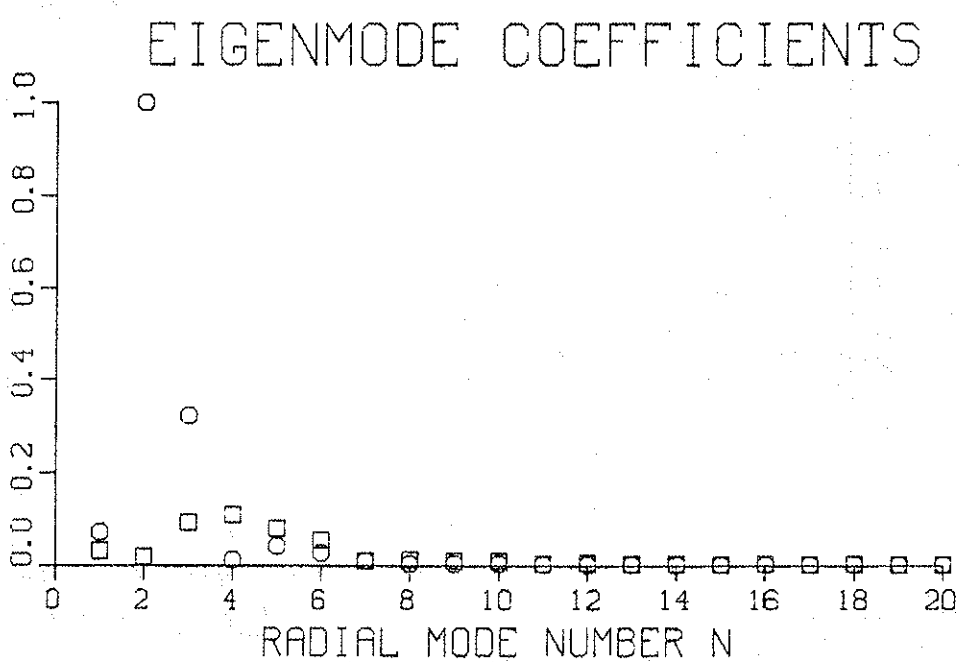
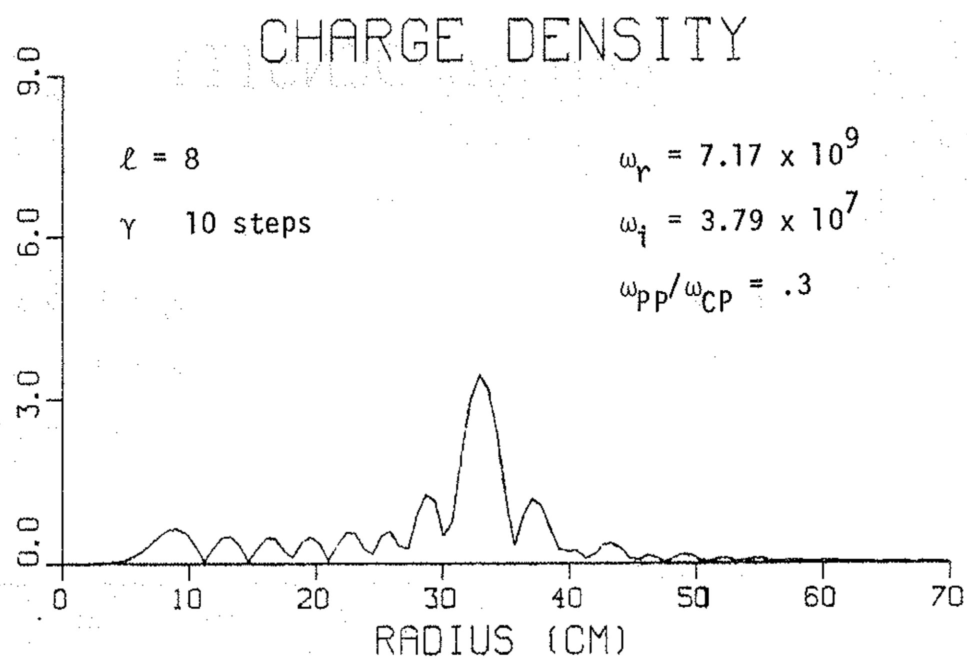

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3. 46


CHARGE DENSITY
9.0
6.0
3.0
0.0
ℓ = 8
γ 10 steps
ωr = 7.17 x 10^9
ωi = 3.79 x 10^7
ωpp/ωcp = .3
0
10
20
30
40
50
60
70
RADIUS (CM)

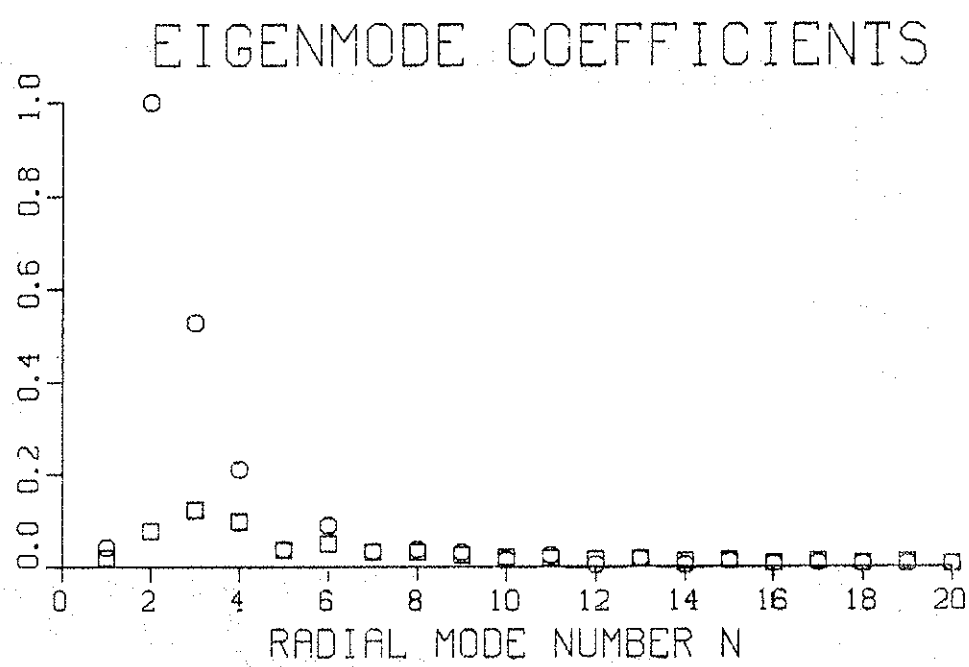
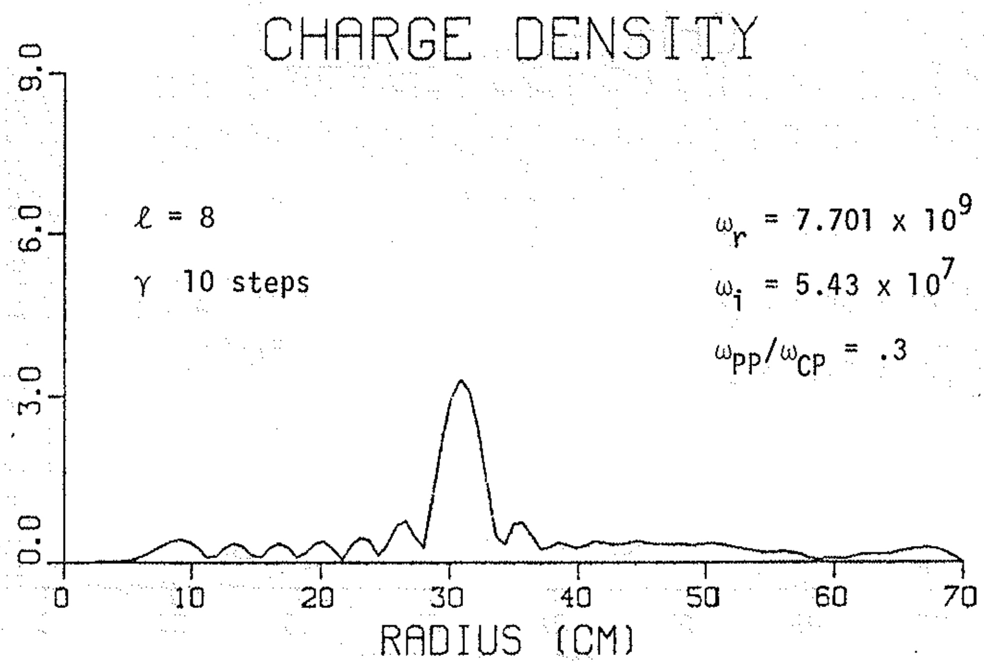

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.47


CHARGE DENSITY
9.0
6.0
3.0
0.0
ℓ = 8
γ 10 steps
ω_r = 7.701 x 10^9
ω_i = 5.43 x 10^7
ω_{pp}/ω_{cp} = .3
0
10
20
30
40
50
60
70
RADIUS (CM)

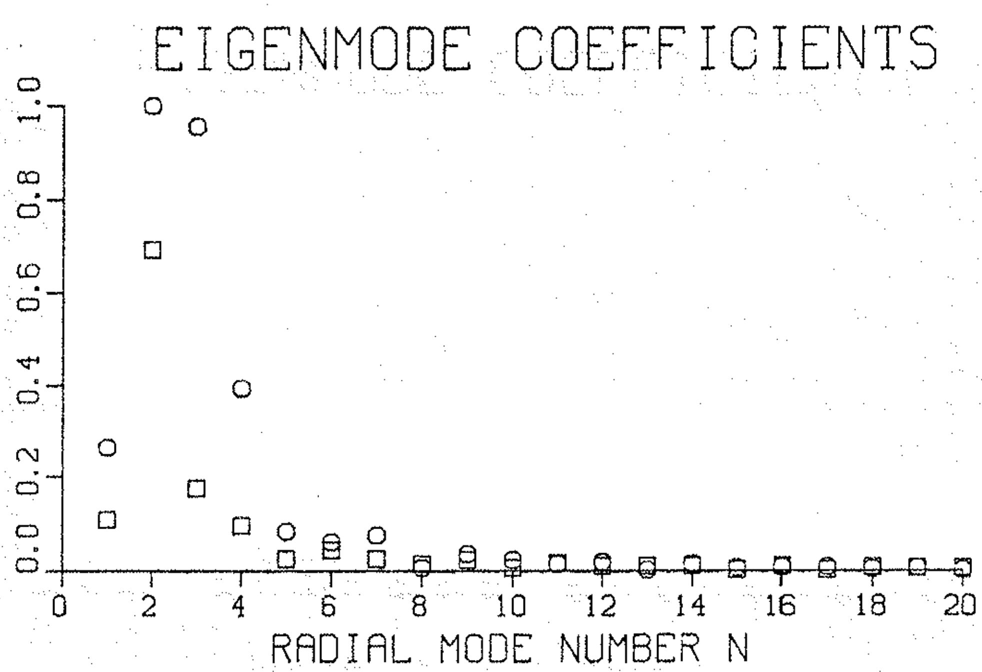
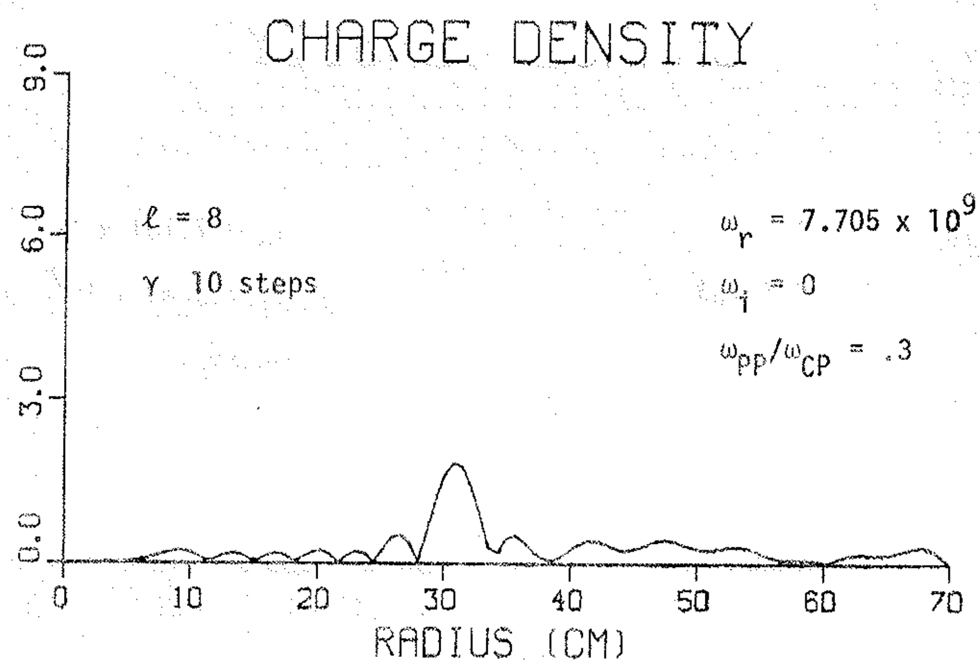

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.48


CHARGE DENSITY
9.0
6.0
3.0
0.0
ℓ = 8
γ 10 steps
ω_r = 7.705 x 10^9
ω_i = 0
ω_pp/ω_CP = .3
0
10
20
30
40
50
60
70
RADIUS (CM)

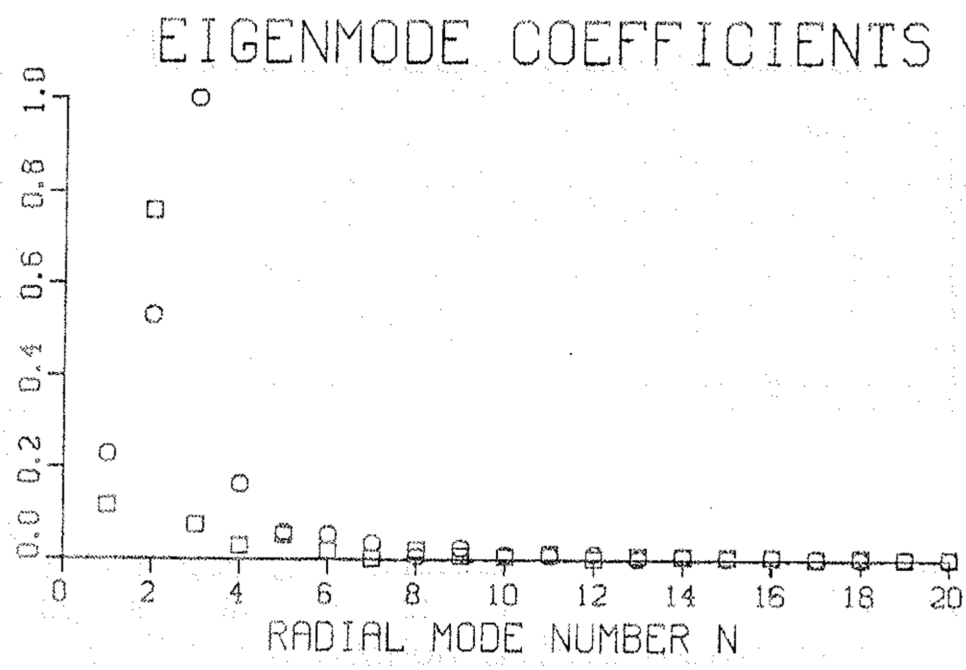
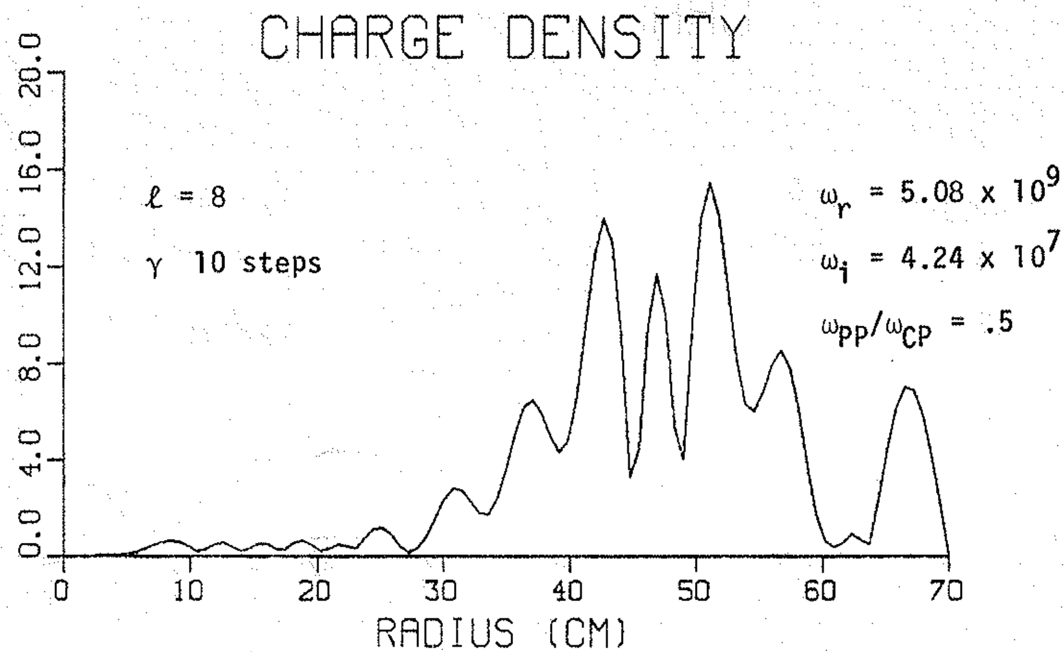

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.49


CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 5.08 \times 10^9$
$\omega_i = 4.24 \times 10^7$
ω_{pp}/ω_{cp} = .5
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)

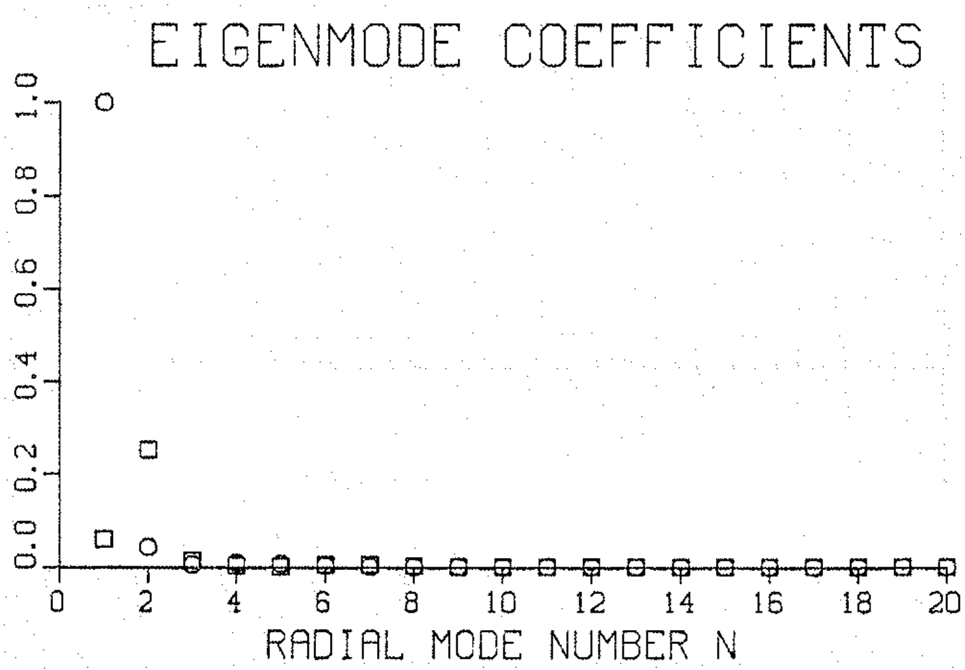

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.50

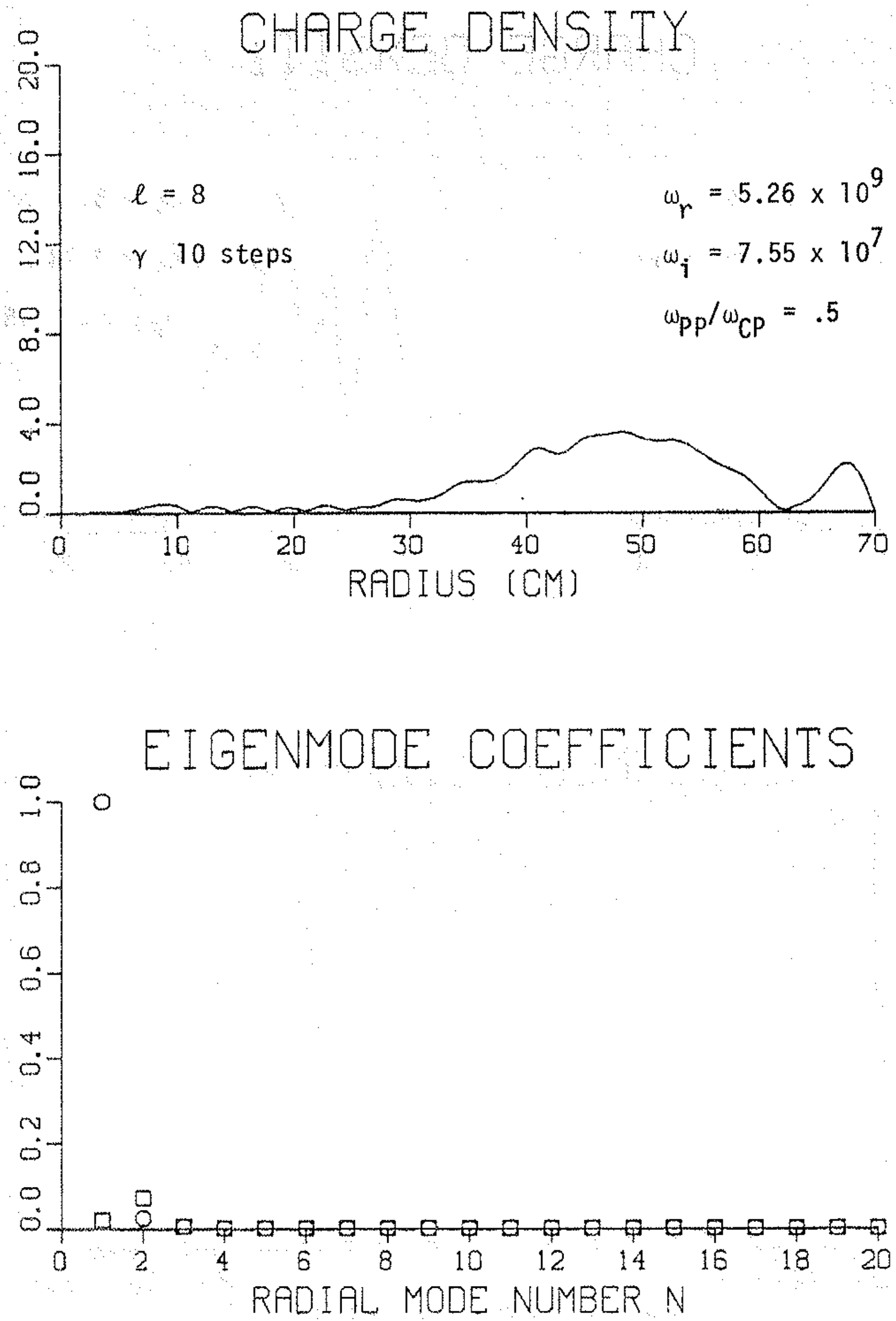

CHARGE DENSITY
ℓ = 8
γ 10 steps
ωr = 5.26 x 10^9
ωi = 7.55 x 10^7
ωpp/ωCP = .5
0.0
4.0
8.0
12.0
16.0
20.0
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.51

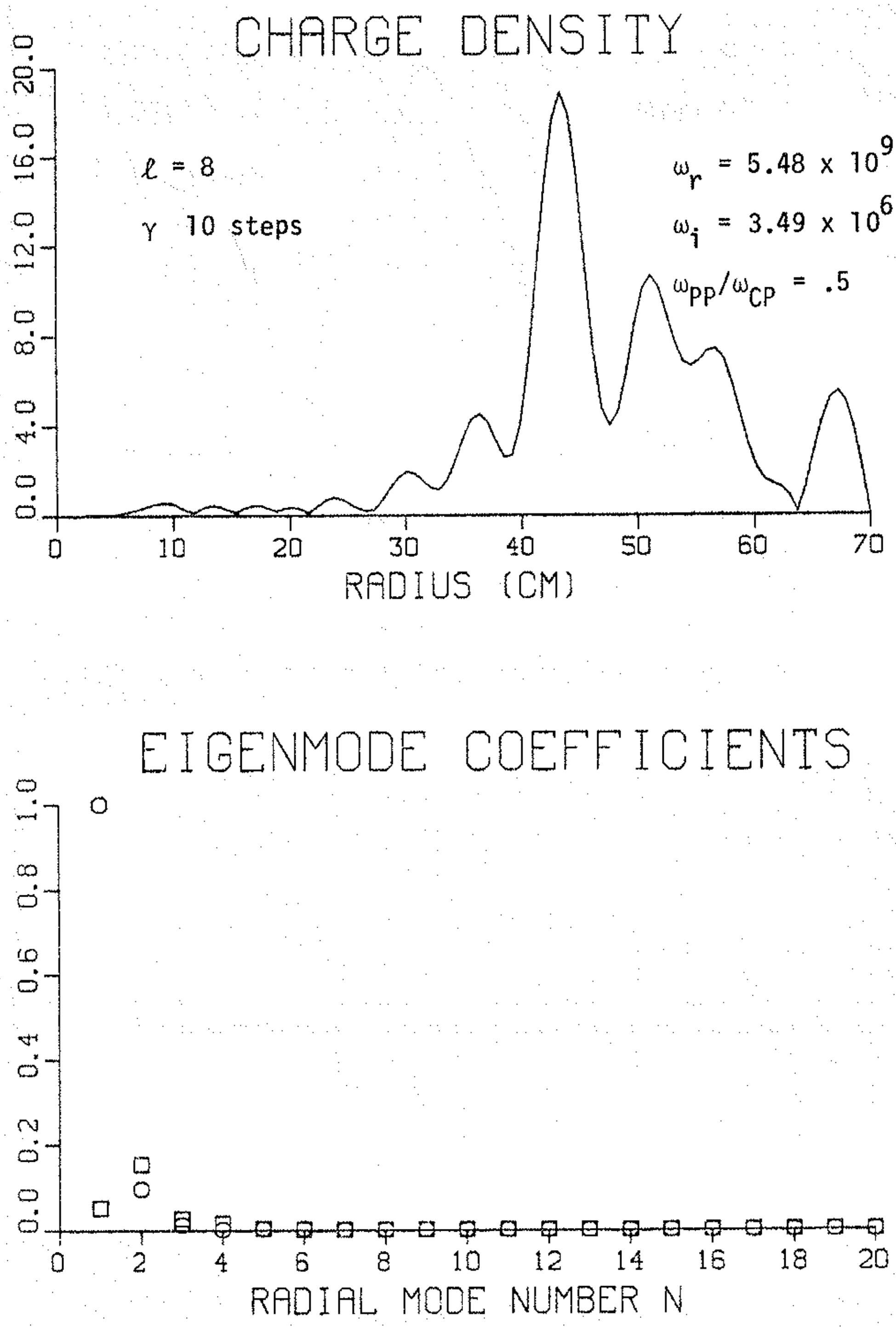
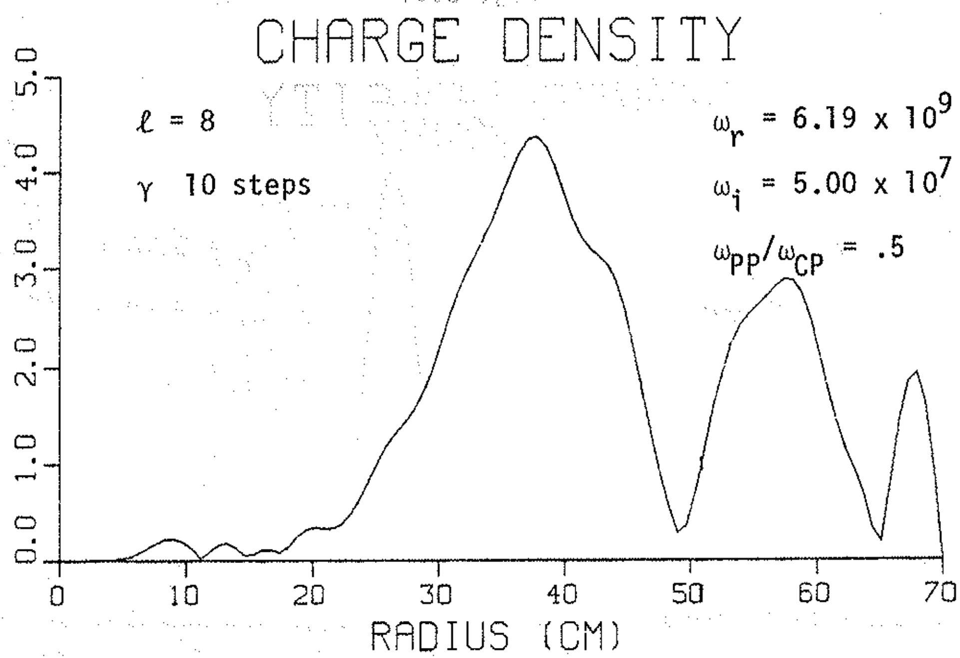
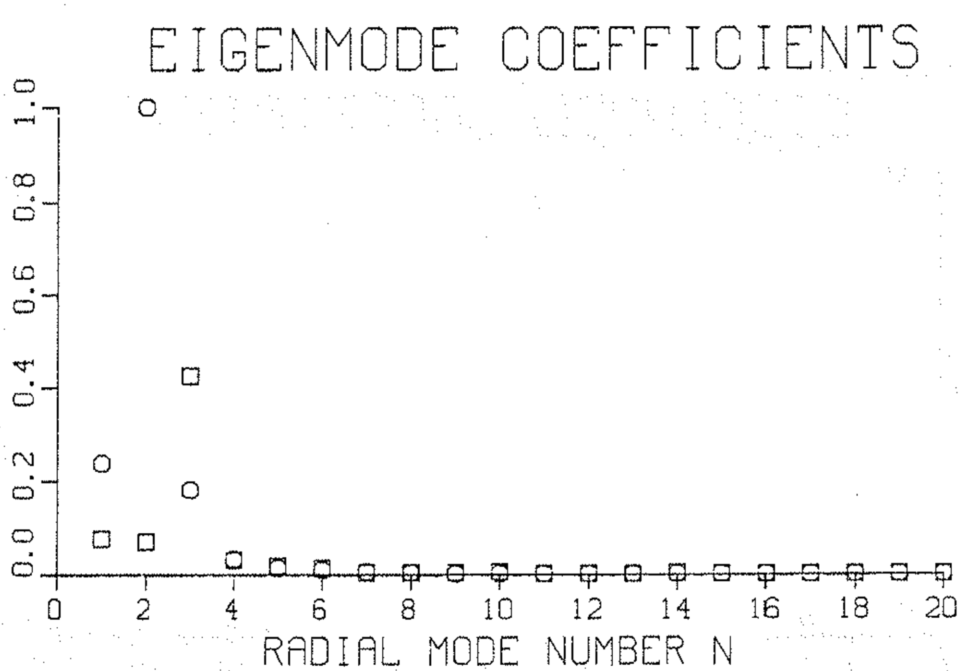

CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 5.48 \times 10^9$
$\omega_i = 3.49 \times 10^6$
$\omega_{PP}/\omega_{CP} = .5$
0.0 4.0 8.0 12.0 16.0 20.0
0 10 20 30 40 50 60 70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 2 4 6 8 10 12 14 16 18 20
RADIAL MODE NUMBER N

Fig. 3.52

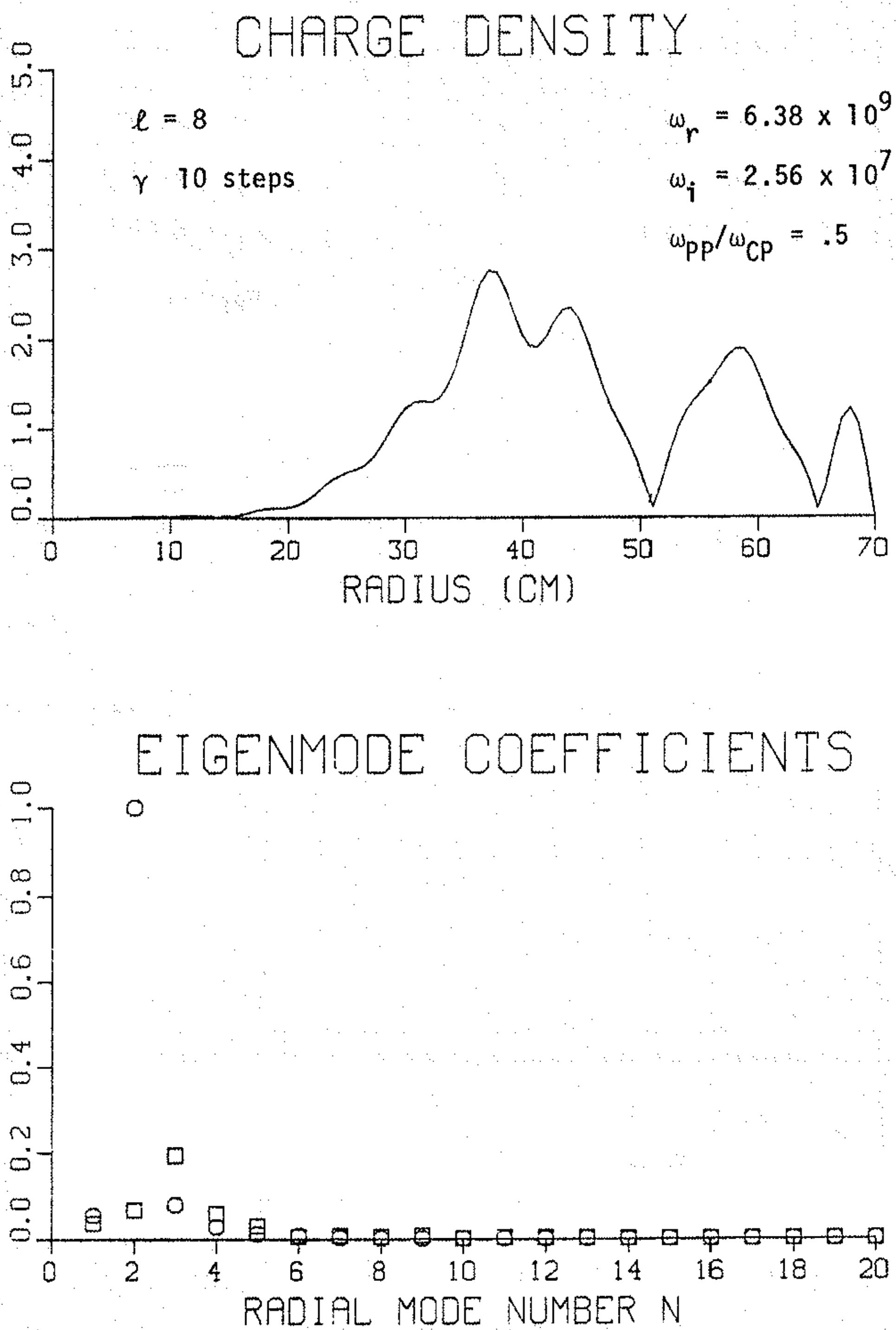


CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 6.19 \times 10^9$
$\omega_i = 5.00 \times 10^7$
$\omega_{PP}/\omega_{CP} = .5$
0.0
1.0
2.0
3.0
4.0
5.0
0
10
20
30
40
50
60
70
RADIUS (CM)



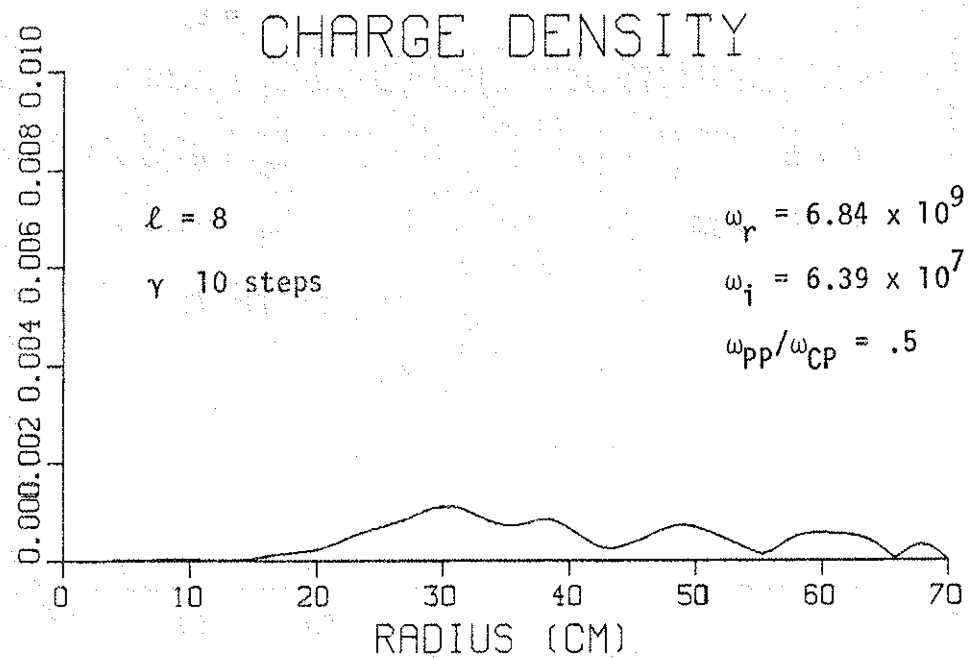
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.53



CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 6.38 x 10^9
ω_i = 2.56 x 10^7
ω_{pp}/ω_{cp} = .5
0.0
1.0
2.0
3.0
4.0
5.0
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.54


CHARGE DENSITY
0.000 0.002 0.004 0.006 0.008 0.010
ℓ = 8
γ 10 steps
ω_r = 6.84 x 10^9
ω_i = 6.39 x 10^7
ω_PP/ω_CP = .5
0 10 20 30 40 50 60 70
RADIUS (CM)

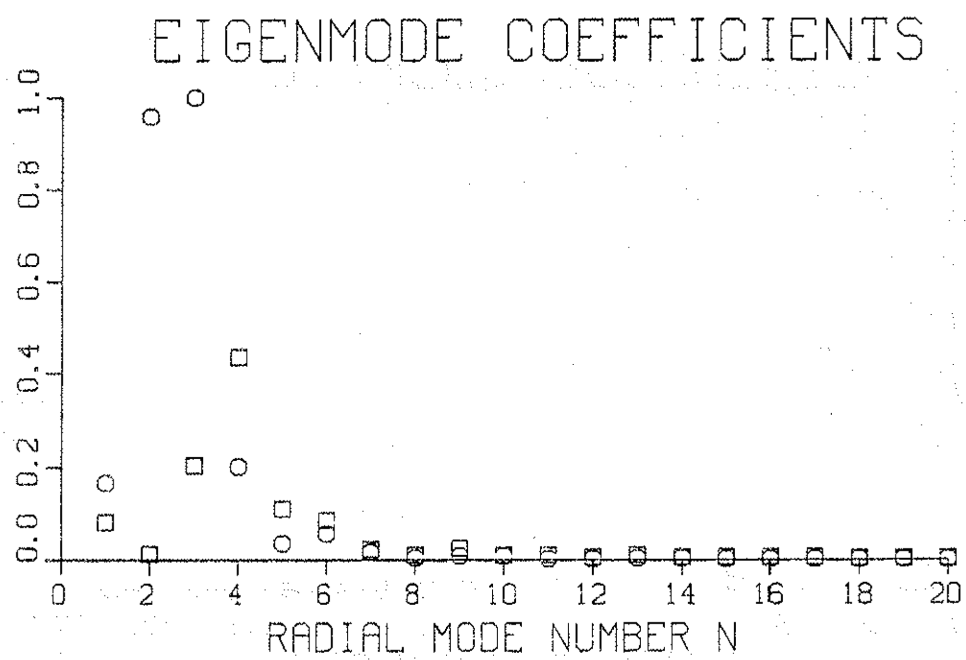
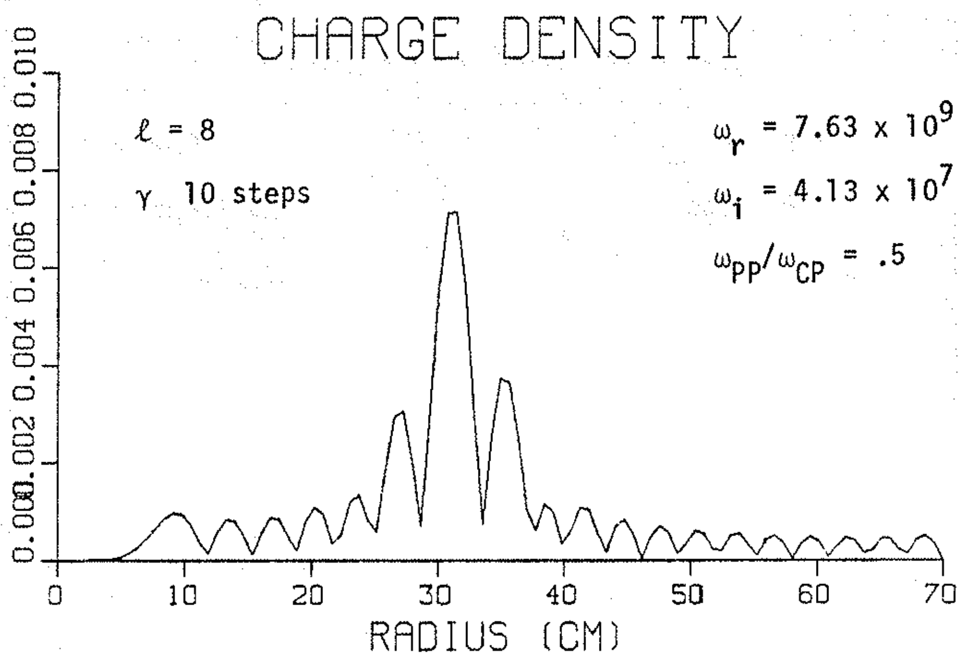

EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 2 4 6 8 10 12 14 16 18 20
RADIAL MODE NUMBER N

Fig. 3.55


CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 7.63 \times 10^9$
$\omega_i = 4.13 \times 10^7$
$\omega_{pp}/\omega_{CP} = .5$
0.000 0.002 0.004 0.006 0.008 0.010
0 10 20 30 40 50 60 70
RADIUS (CM)

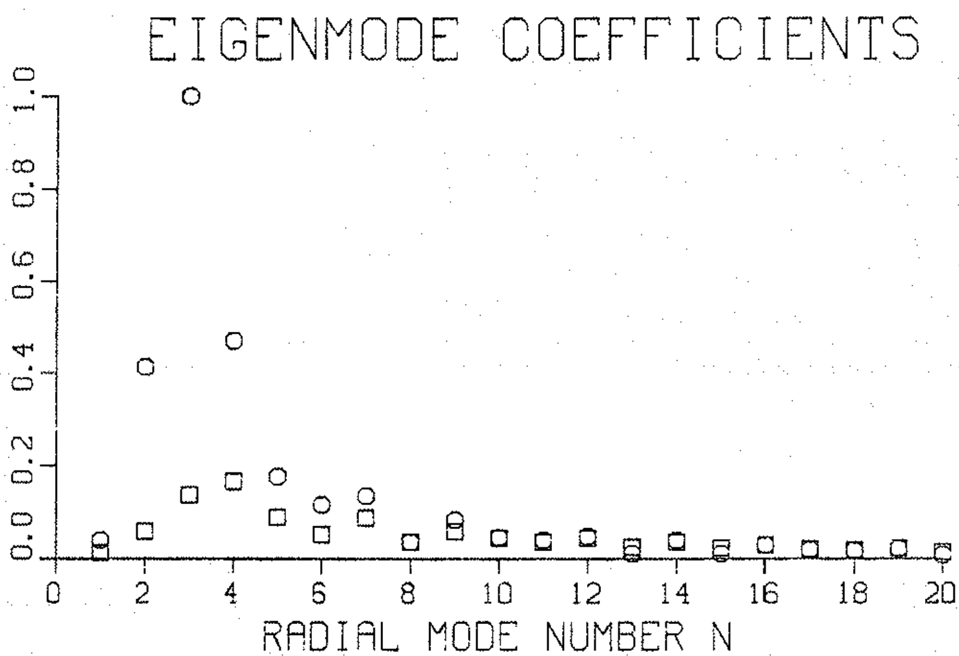
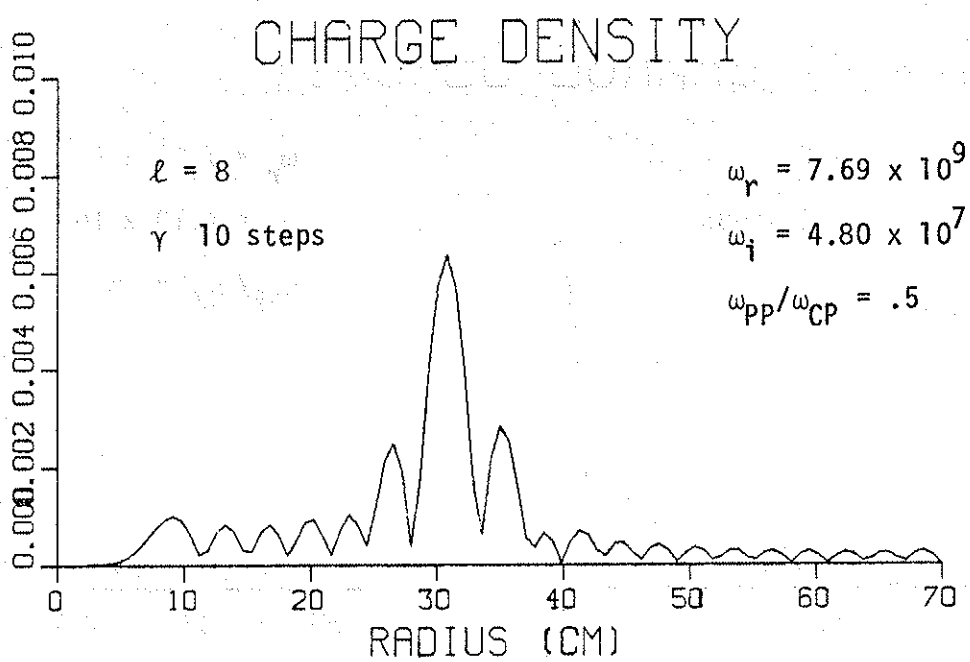

EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 2 4 6 8 10 12 14 16 18 20
RADIAL MODE NUMBER N

Fig. 3.56


CHARGE DENSITY
ℓ = 8
γ 10 steps
ω_r = 7.69 x 10^9
ω_i = 4.80 x 10^7
ω_{PP}/ω_{CP} = .5
0.000 0.002 0.004 0.006 0.008 0.010
0 10 20 30 40 50 60 70
RADIUS (CM)

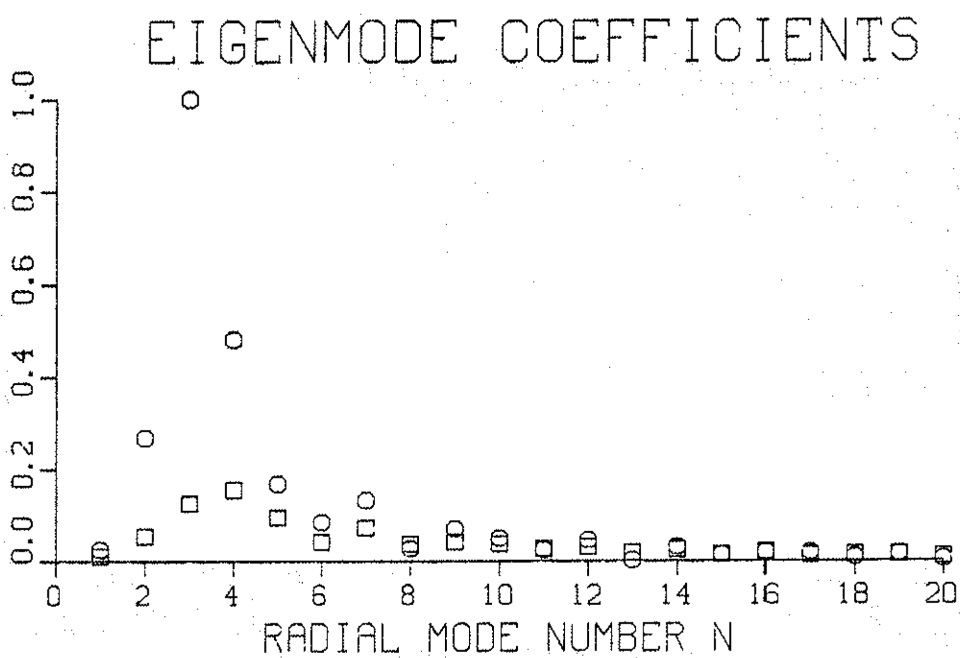

EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 2 4 6 8 10 12 14 16 18 20
RADIAL MODE NUMBER N

Fig. 3.57

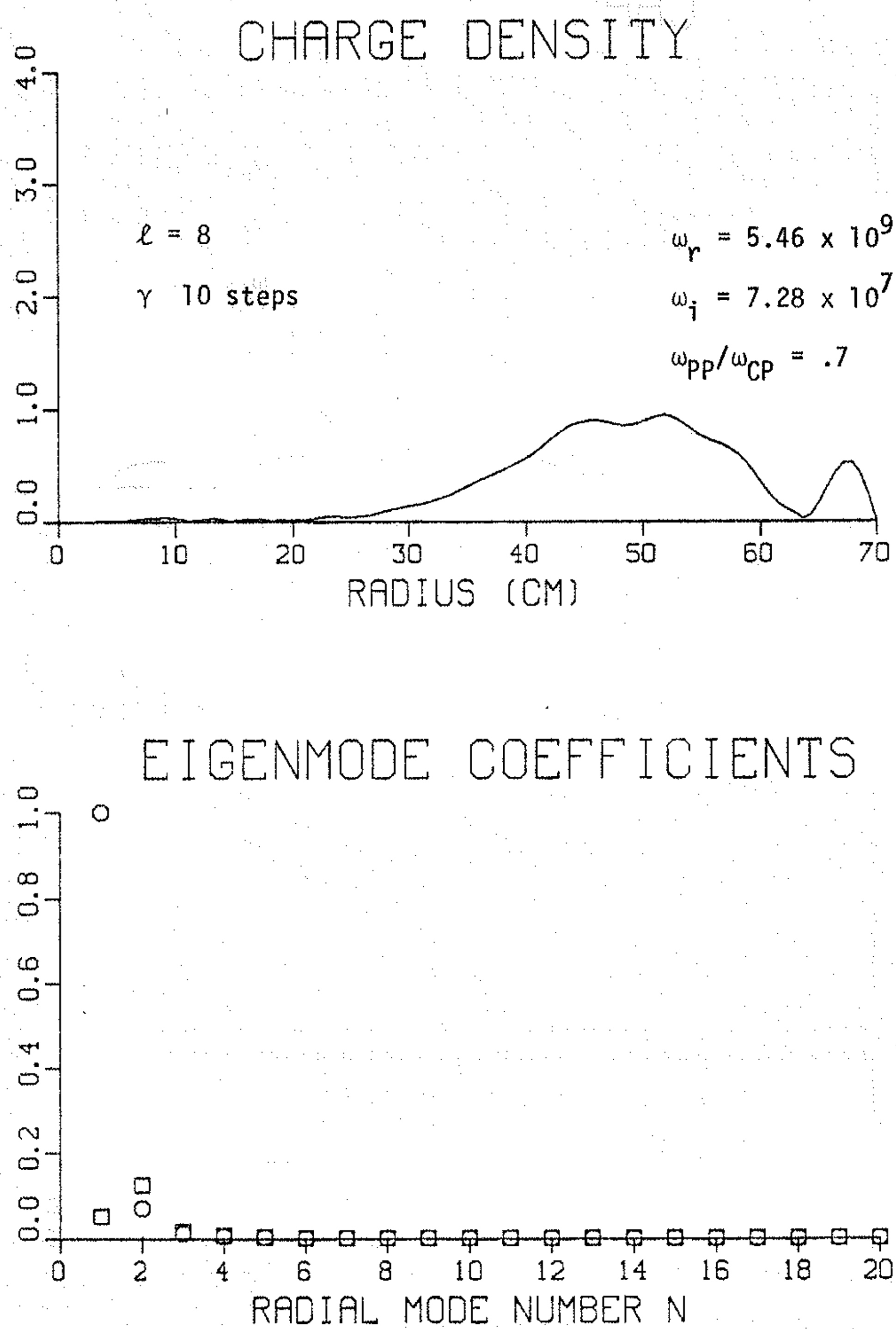
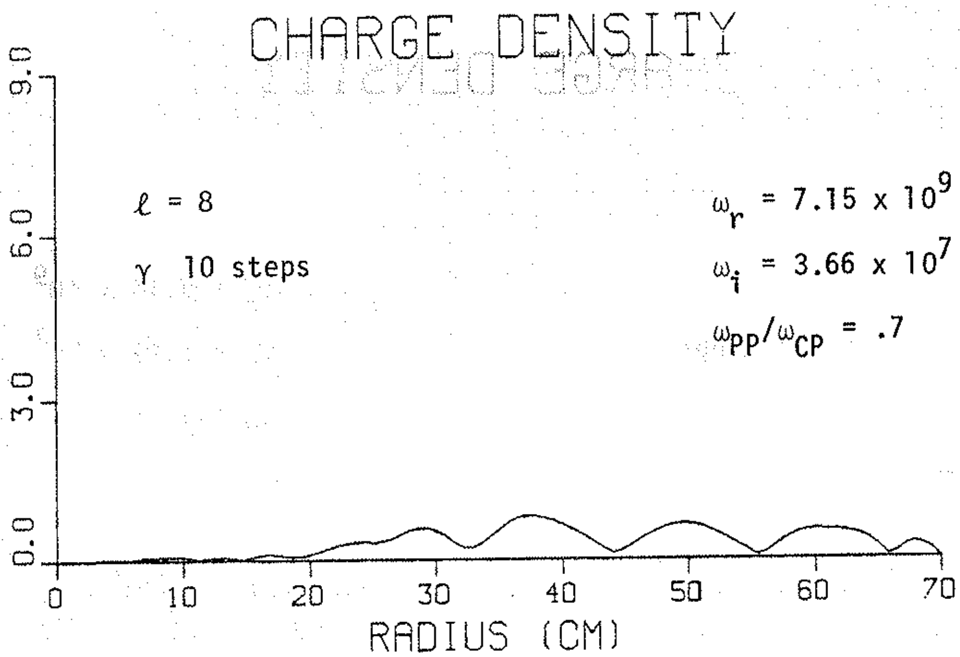

CHARGE DENSITY
ℓ = 8
γ 10 steps
$\omega_r = 5.46 \times 10^9$
$\omega_i = 7.28 \times 10^7$
$\omega_{PP}/\omega_{CP} = .7$
0.0
1.0
2.0
3.0
4.0
0
10
20
30
40
50
60
70
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.58


CHARGE DENSITY
9.0
6.0
3.0
0.0
ℓ = 8
γ 10 steps
ω_r = 7.15 x 10^9
ω_i = 3.66 x 10^7
ω_{PP}/ω_{CP} = .7
0
10
20
30
40
50
60
70
RADIUS (CM)

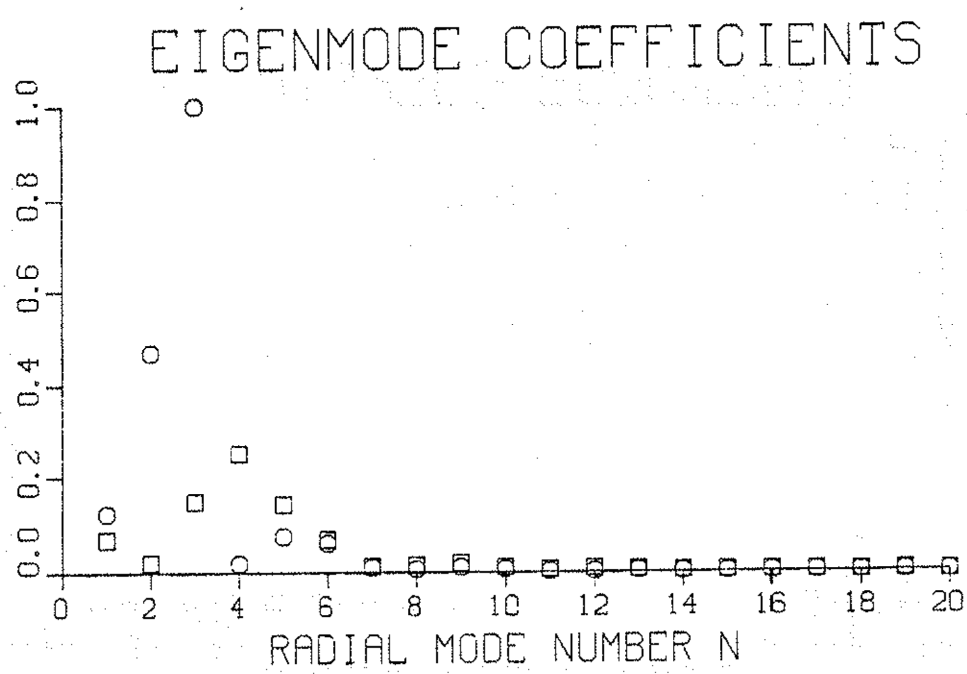
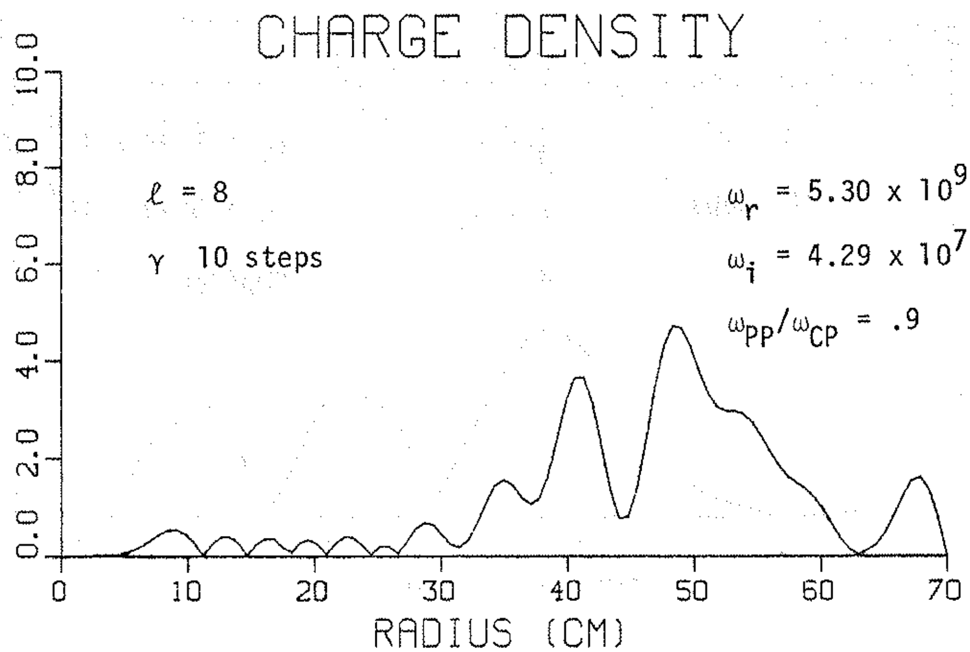

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

Fig. 3.59


CHARGE DENSITY
10.0
8.0
6.0
4.0
2.0
0.0
ℓ = 8
γ 10 steps
ω_r = 5.30 x 10^9
ω_i = 4.29 x 10^7
ω_{PP}/ω_{CP} = .9
0
10
20
30
40
50
60
70
RADIUS (CM)

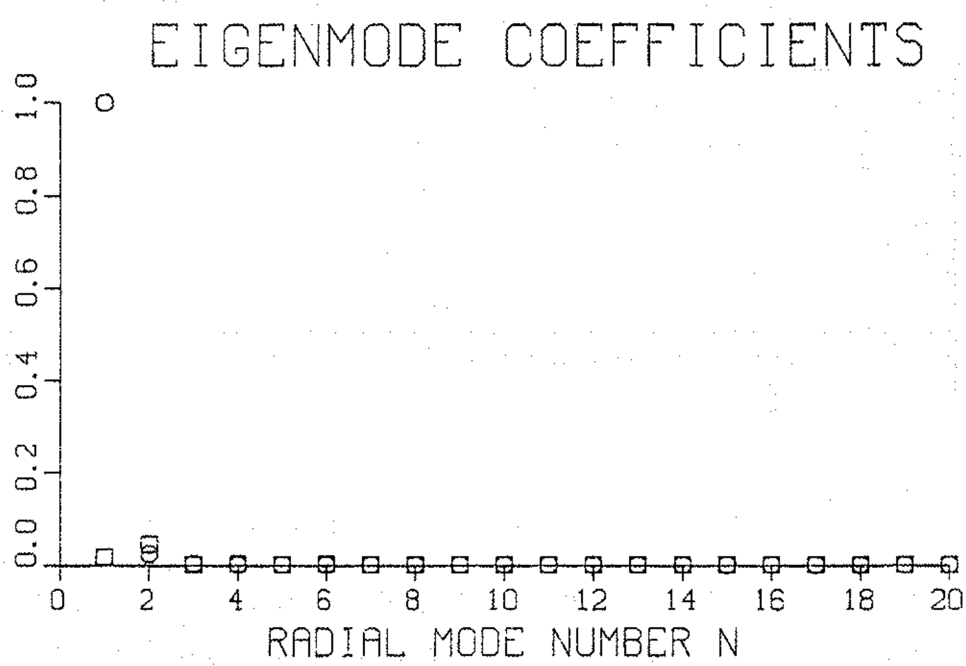
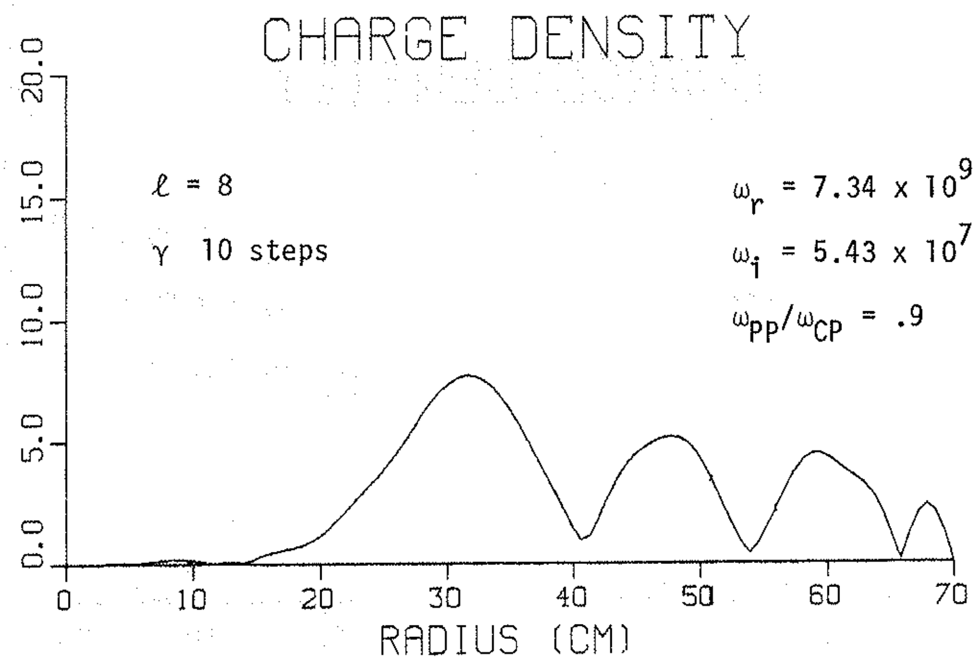
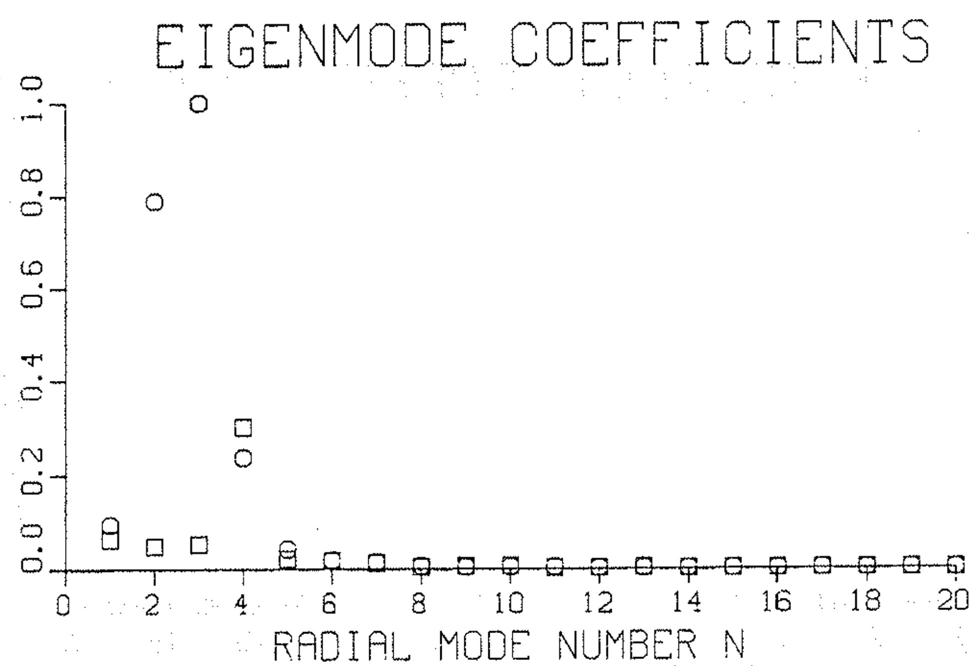
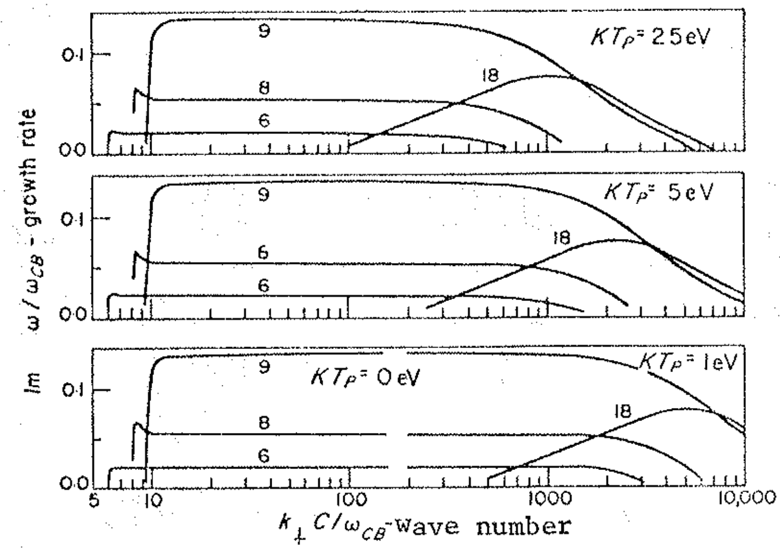

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N

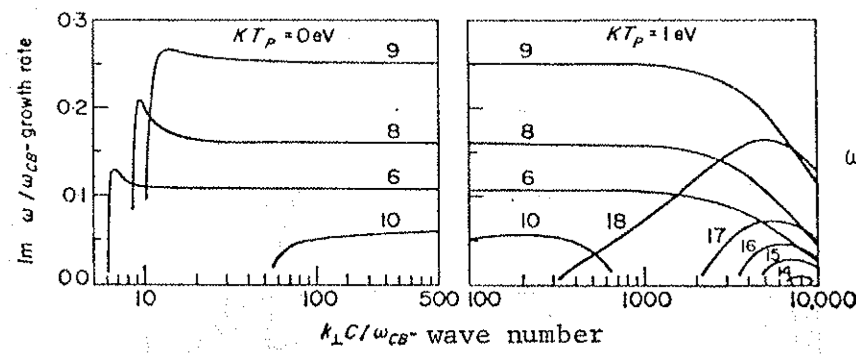
Fig. 3.60


CHARGE DENSITY
20.0
15.0
10.0
5.0
0.0
ℓ = 8
γ 10 steps
$\omega_r = 7.34 \times 10^9$
$\omega_i = 5.43 \times 10^7$
$\omega_{PP}/\omega_{CP} = .9$
0
10
20
30
40
50
60
70
RADIUS (CM)

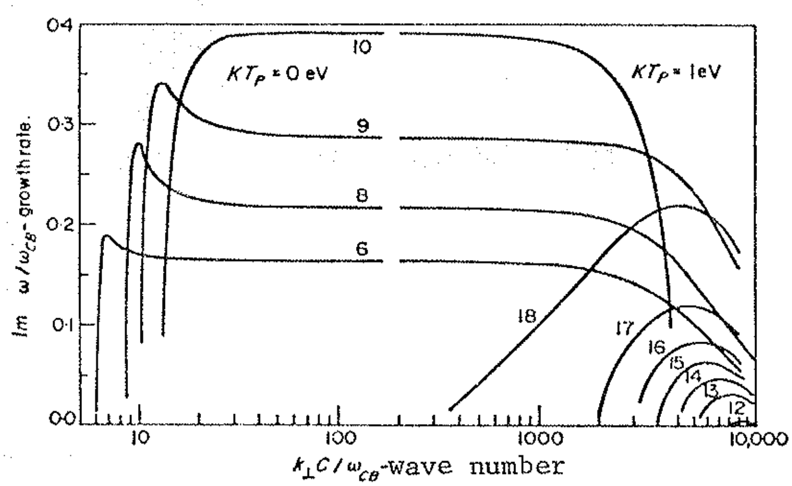

EIGENMODE COEFFICIENTS
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
RADIAL MODE NUMBER N



ω_{PP}/ω_{CP} = .1

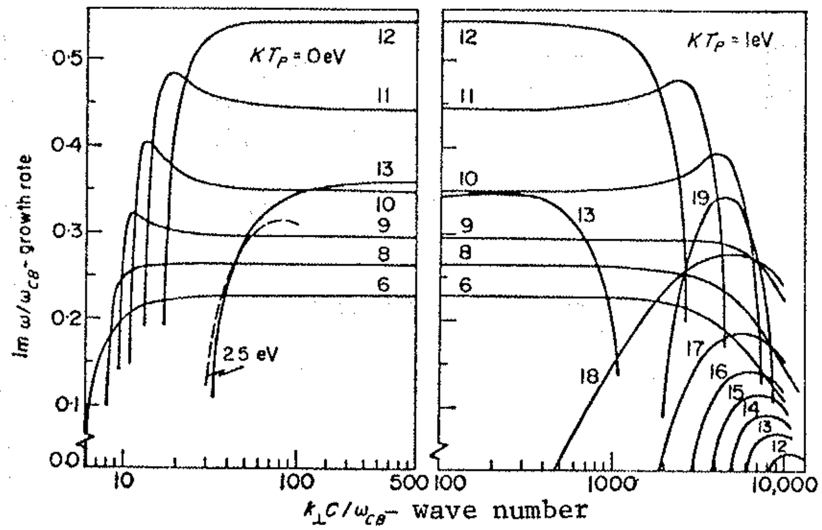


ω_{PP}/ω_{CP} = .3

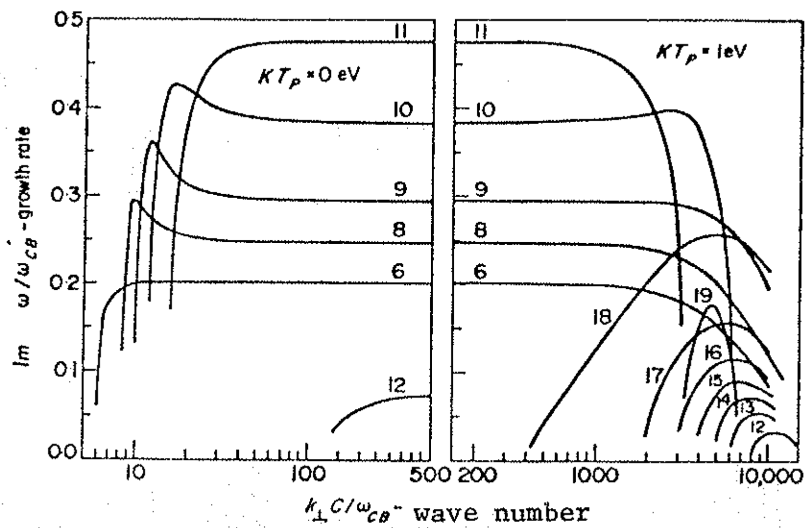


ω_{PP}/ω_{CP} = .5

Fig. 3.61a Reproduced from Ref. 12.



ω_{PP}/ω_{CP} = .7



ω_{PP}/ω_{CP} = .9

Fig. 3.61b

Reproduced from Ref. 12.

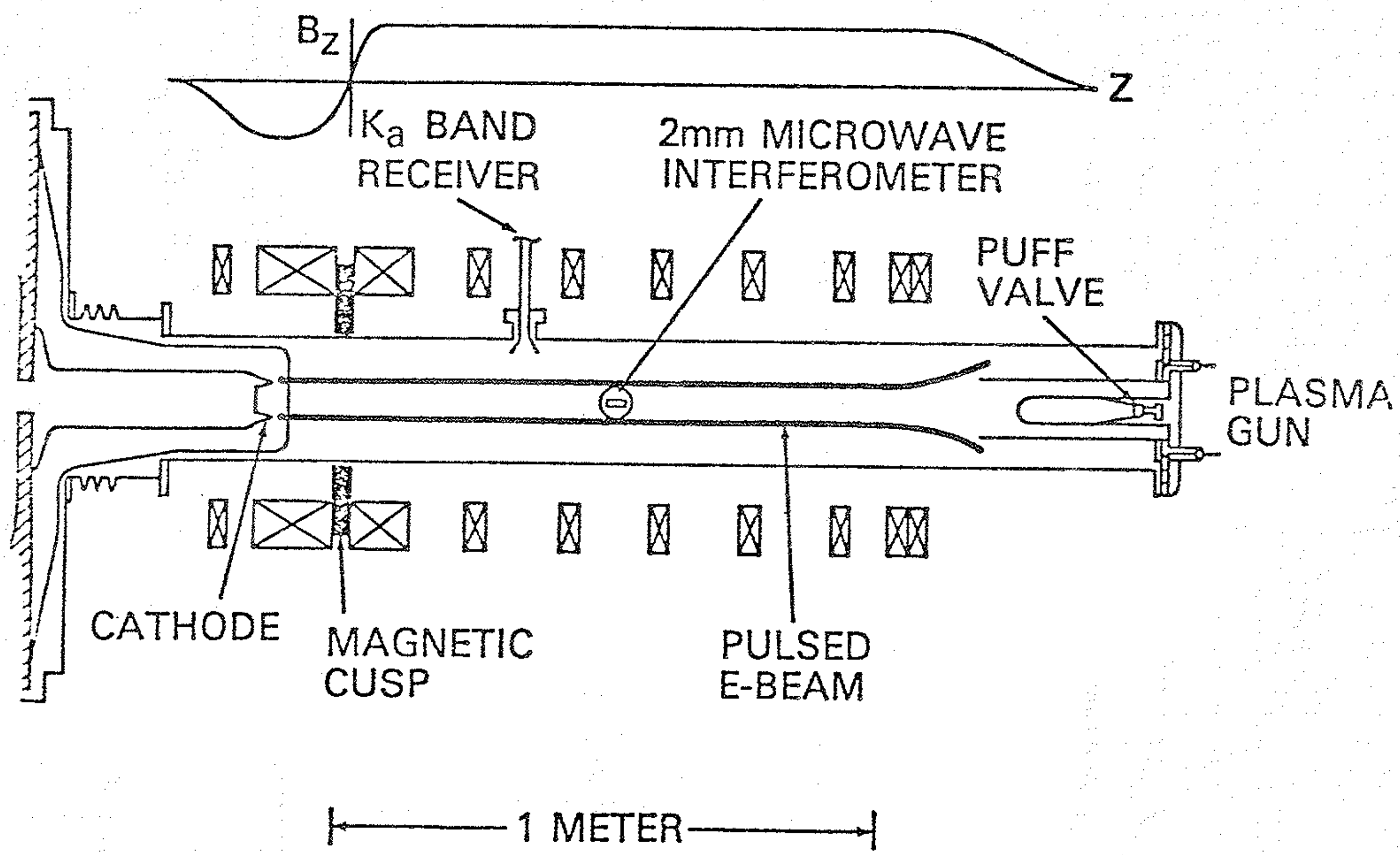
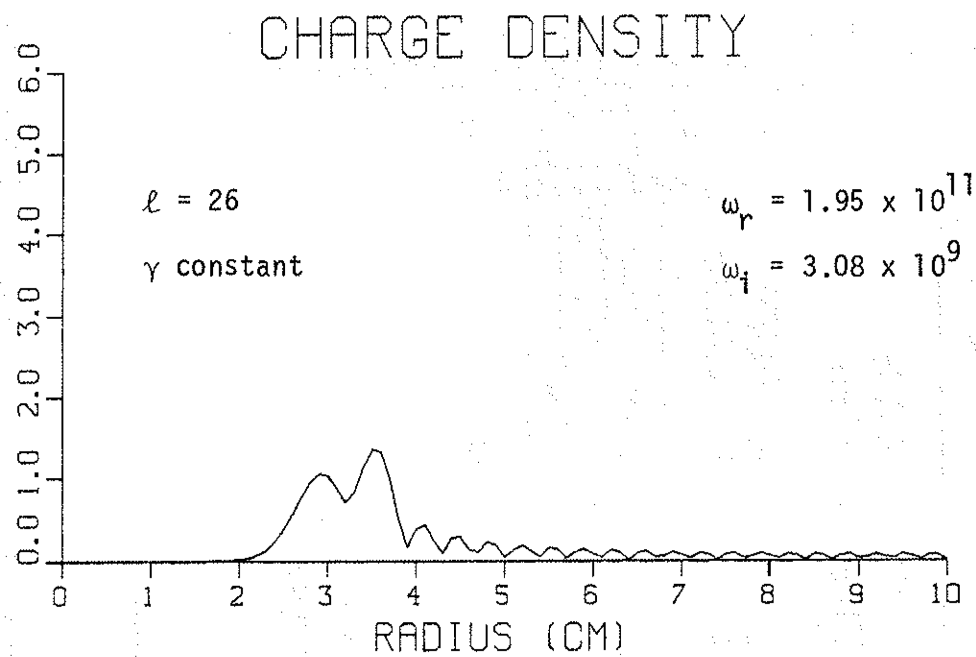


Fig. 3.62 (Reproduced from Ref. 16)

Fig. 3.63


CHARGE DENSITY
ℓ = 26
γ constant
ω_r = 1.95 x 10^{11}
ω_i = 3.08 x 10^9
0.0 1.0 2.0 3.0 4.0 5.0 6.0
0 1 2 3 4 5 6 7 8 9 10
RADIUS (CM)

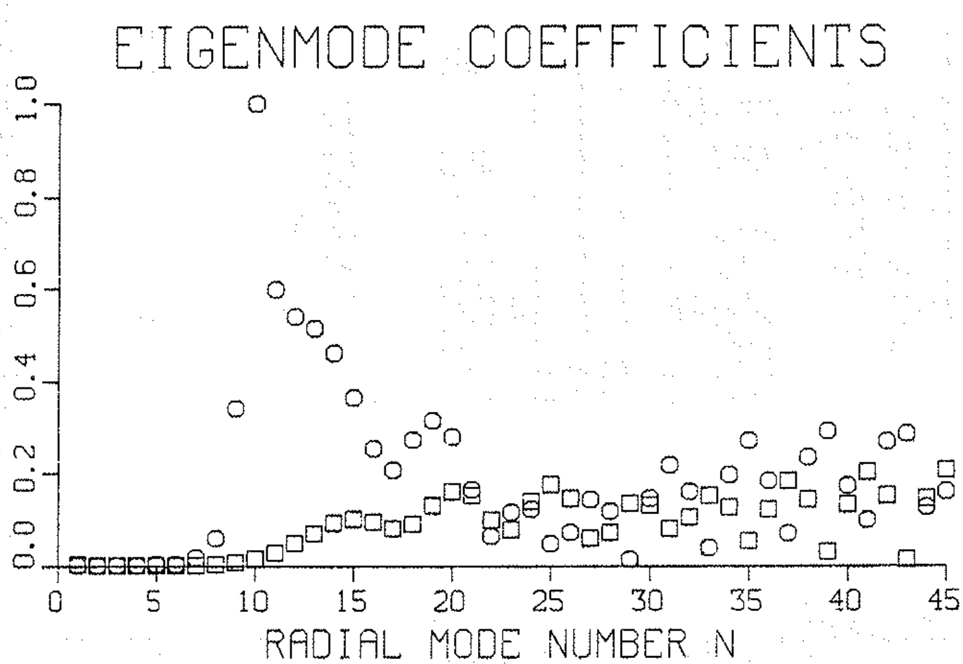
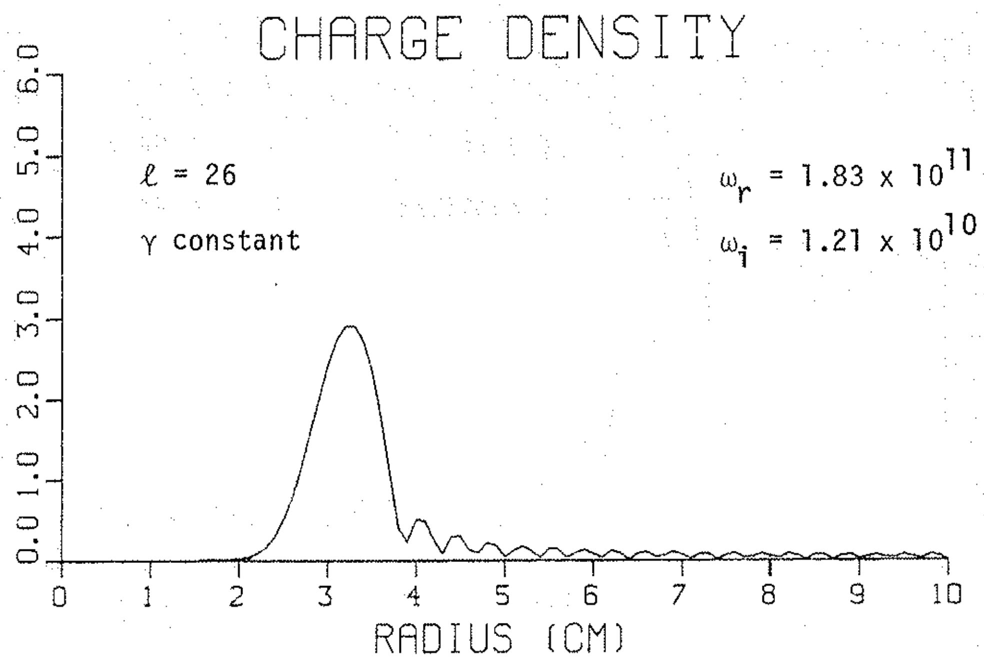

EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 5 10 15 20 25 30 35 40 45
RADIAL MODE NUMBER N

Fig. 3.64


CHARGE DENSITY
ℓ = 26
γ constant
ω_r = 1.83 x 10^{11}
ω_i = 1.21 x 10^{10}
0.0 1.0 2.0 3.0 4.0 5.0 6.0
0 1 2 3 4 5 6 7 8 9 10
RADIUS (CM)

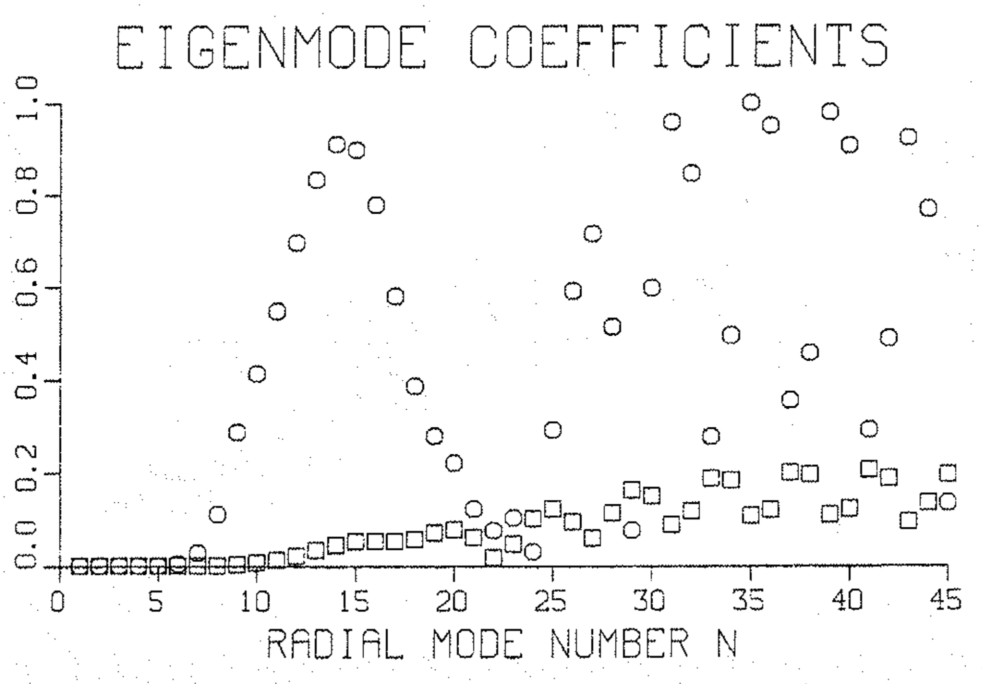
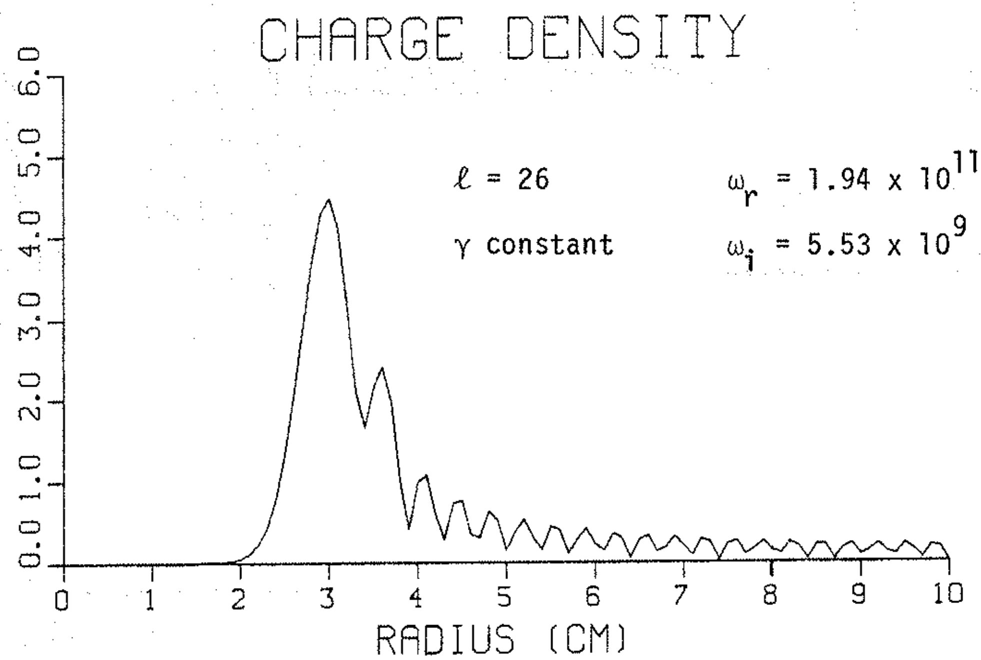

EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 5 10 15 20 25 30 35 40 45
RADIAL MODE NUMBER N

Fig. 3.65


CHARGE DENSITY
ℓ = 26
ω_r = 1.94 x 10^{11}
γ constant
ω_i = 5.53 x 10^9
0.0 1.0 2.0 3.0 4.0 5.0 6.0
0 1 2 3 4 5 6 7 8 9 10
RADIUS (CM)

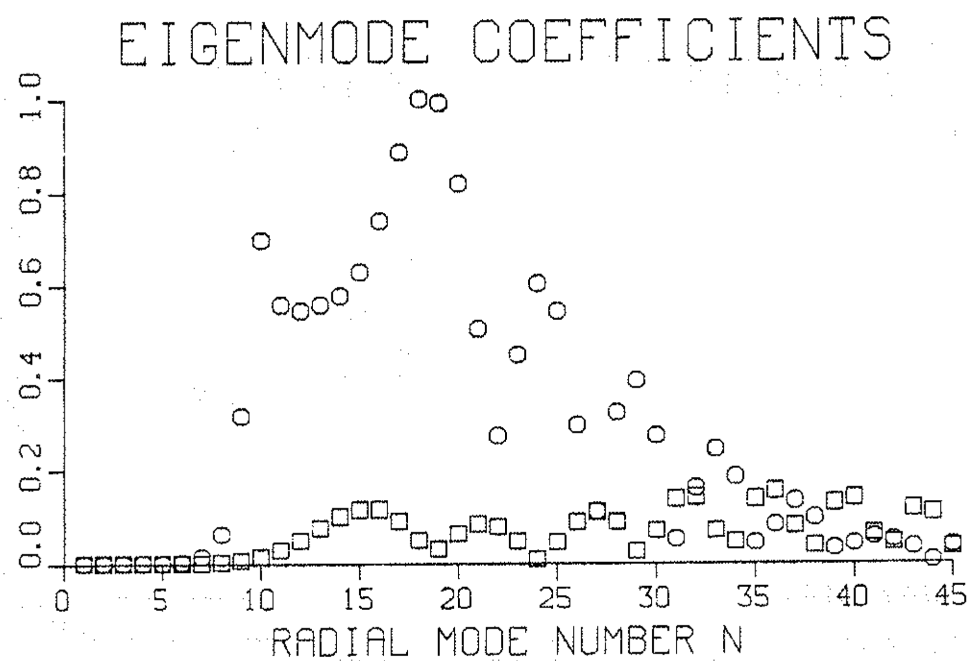

EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 5 10 15 20 25 30 35 40 45
RADIAL MODE NUMBER N

Fig. 3.66

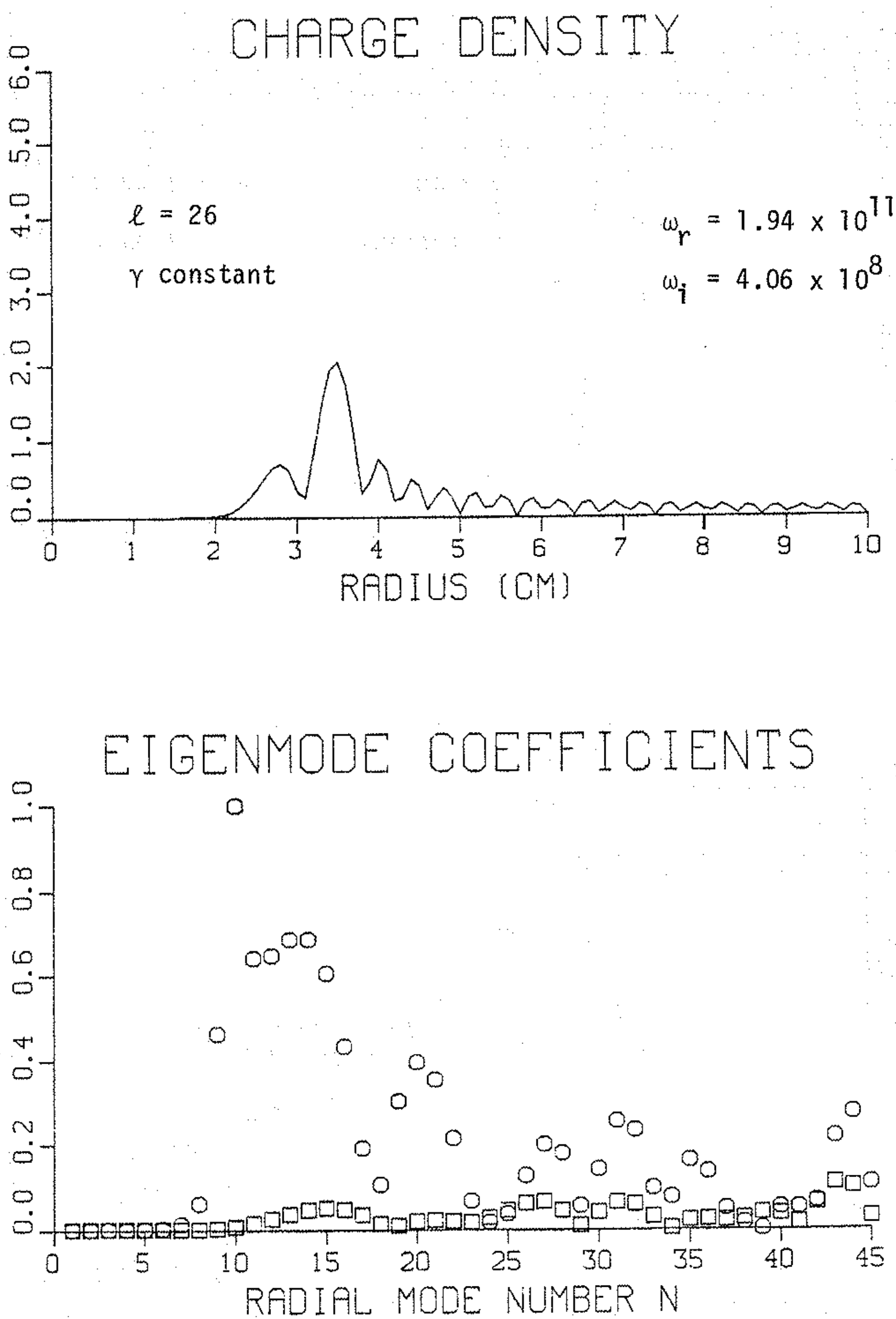
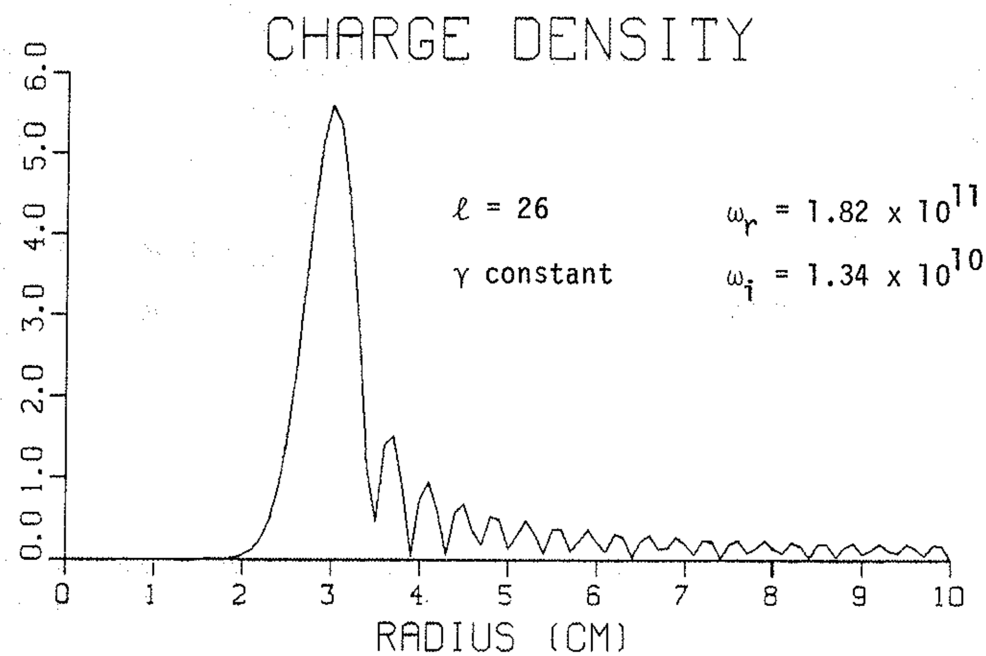

CHARGE DENSITY
0.0
1.0
2.0
3.0
4.0
5.0
6.0
ℓ = 26
γ constant
$\omega_r = 1.94 \times 10^{11}$
$\omega_i = 4.06 \times 10^8$
0
1
2
3
4
5
6
7
8
9
10
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
35
40
45
RADIAL MODE NUMBER N

Fig. 3.67


CHARGE DENSITY
0.0
1.0
2.0
3.0
4.0
5.0
6.0
ℓ = 26
γ constant
$\omega_r = 1.82 \times 10^{11}$
$\omega_i = 1.34 \times 10^{10}$
0
1
2
3
4
5
6
7
8
9
10
RADIUS (CM)

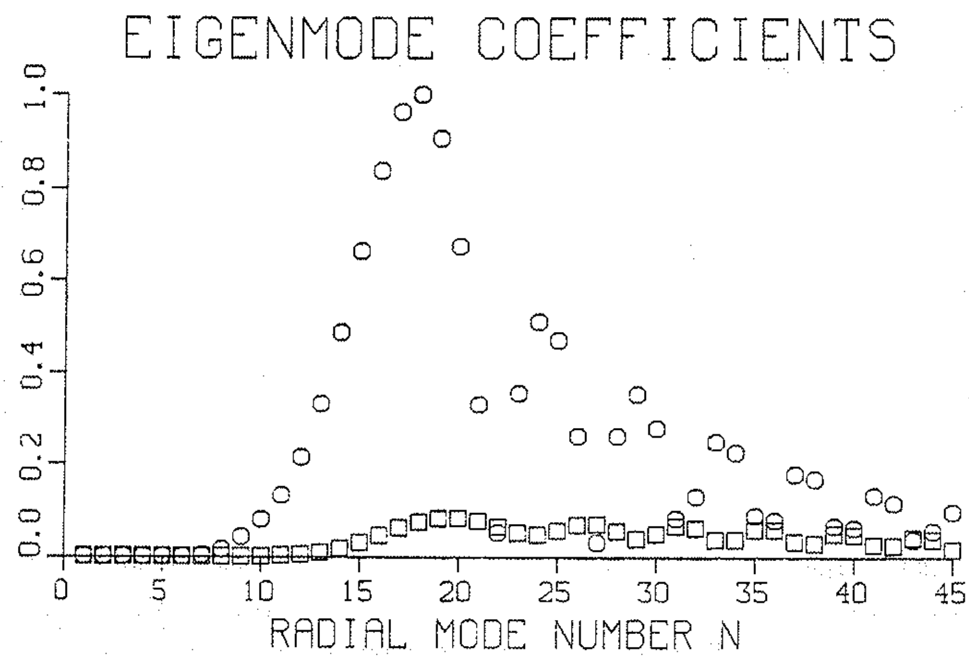

EIGENMODE COEFFICIENTS
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
35
40
45
RADIAL MODE NUMBER N

Fig. 3.68

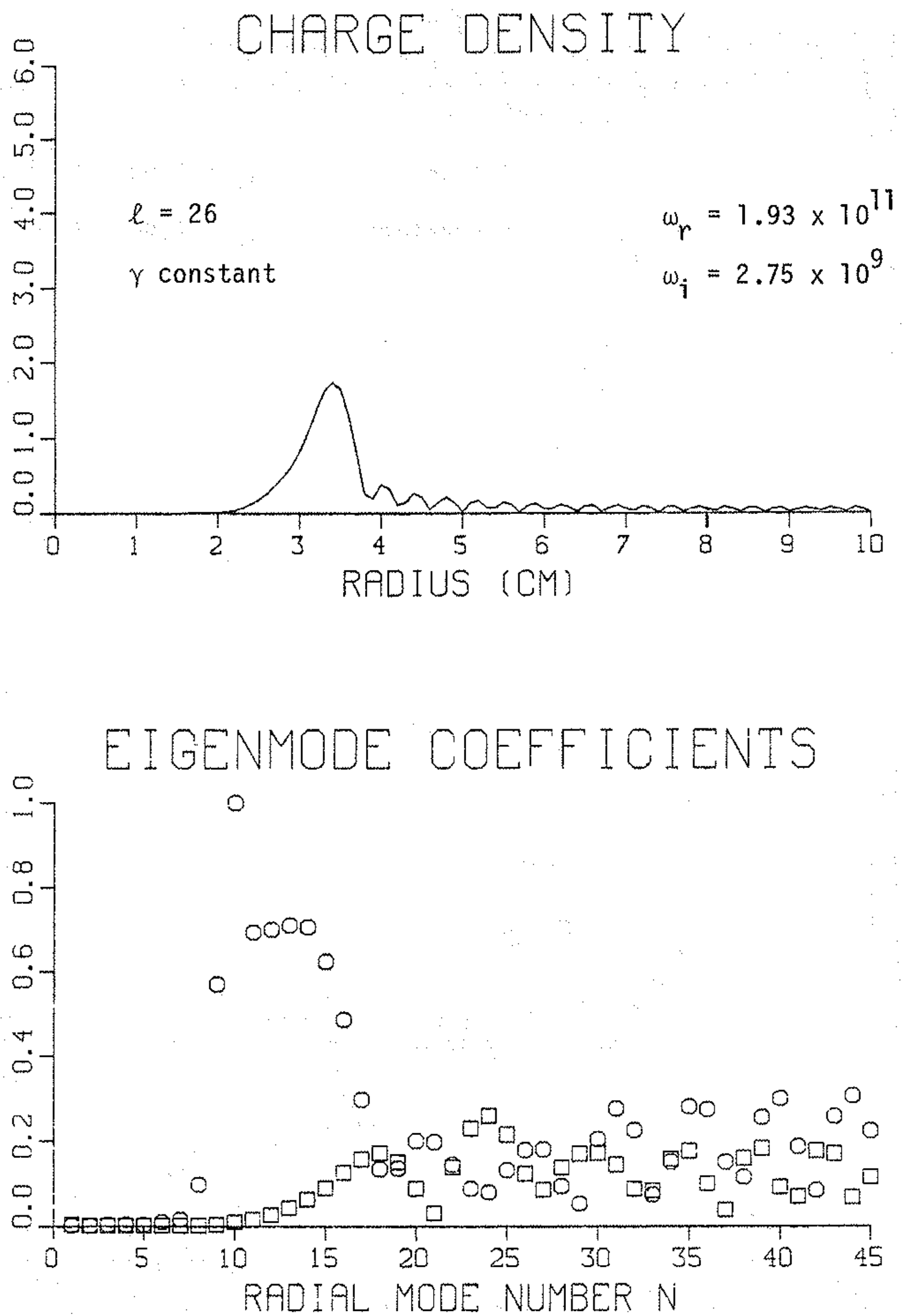

CHARGE DENSITY
ℓ = 26
γ constant
$\omega_r = 1.93 \times 10^{11}$
$\omega_i = 2.75 \times 10^{9}$
RADIUS (CM)
EIGENMODE COEFFICIENTS
RADIAL MODE NUMBER N

Fig. 3.69

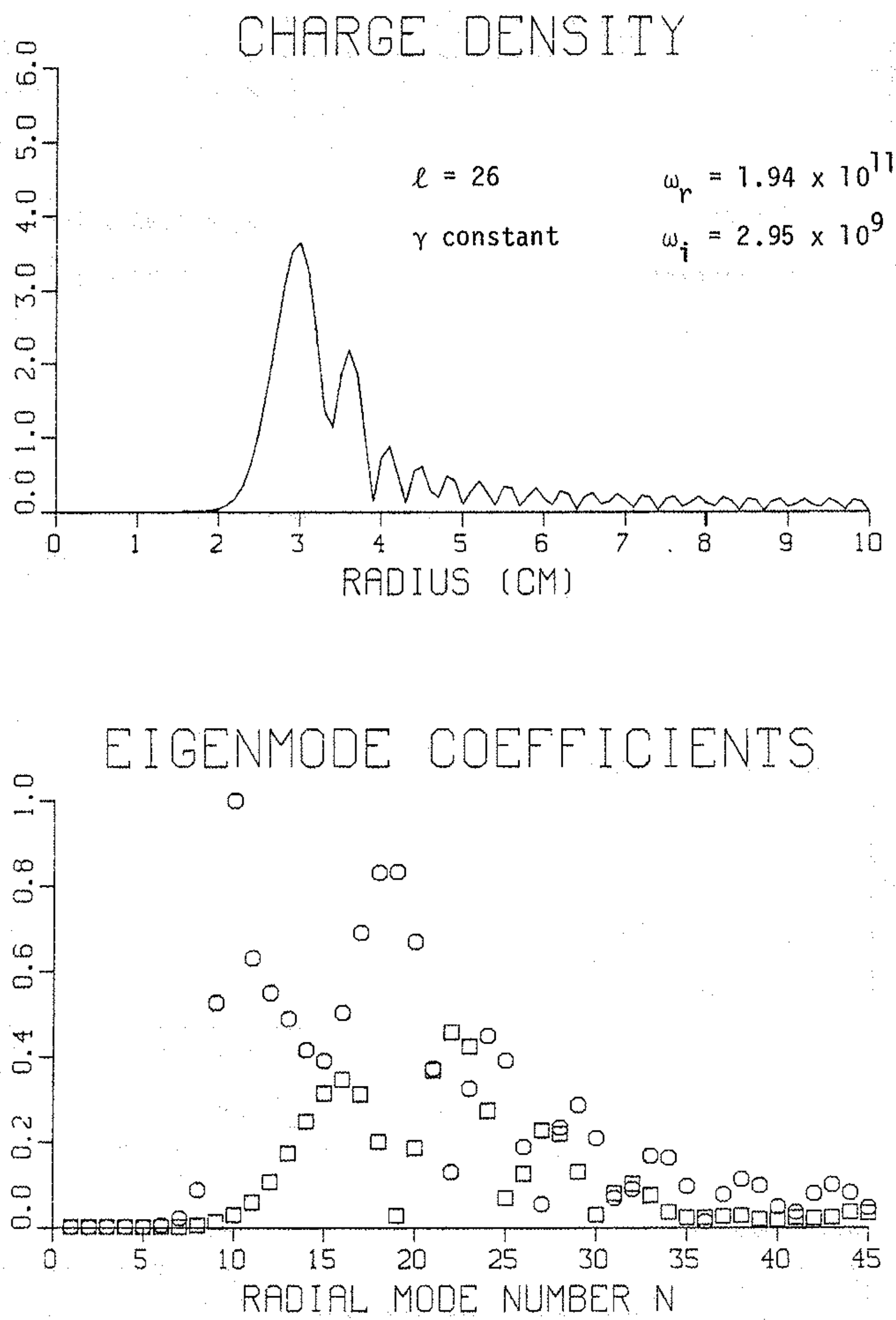
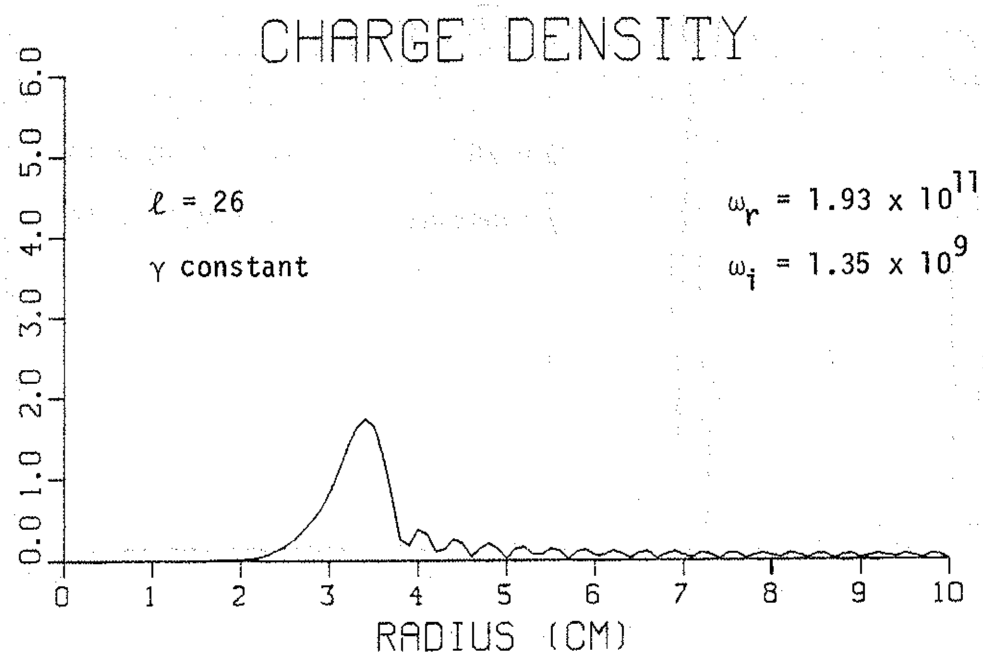

CHARGE DENSITY
ℓ = 26
γ constant
$\omega_r = 1.94 \times 10^{11}$
$\omega_i = 2.95 \times 10^{9}$
0.0 1.0 2.0 3.0 4.0 5.0 6.0
0 1 2 3 4 5 6 7 8 9 10
RADIUS (CM)
EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 5 10 15 20 25 30 35 40 45
RADIAL MODE NUMBER N

Fig. 3.70


CHARGE DENSITY
ℓ = 26
γ constant
ω_r = 1.93 x 10^{11}
ω_i = 1.35 x 10^9
0.0 1.0 2.0 3.0 4.0 5.0 6.0
0 1 2 3 4 5 6 7 8 9 10
RADIUS (CM)

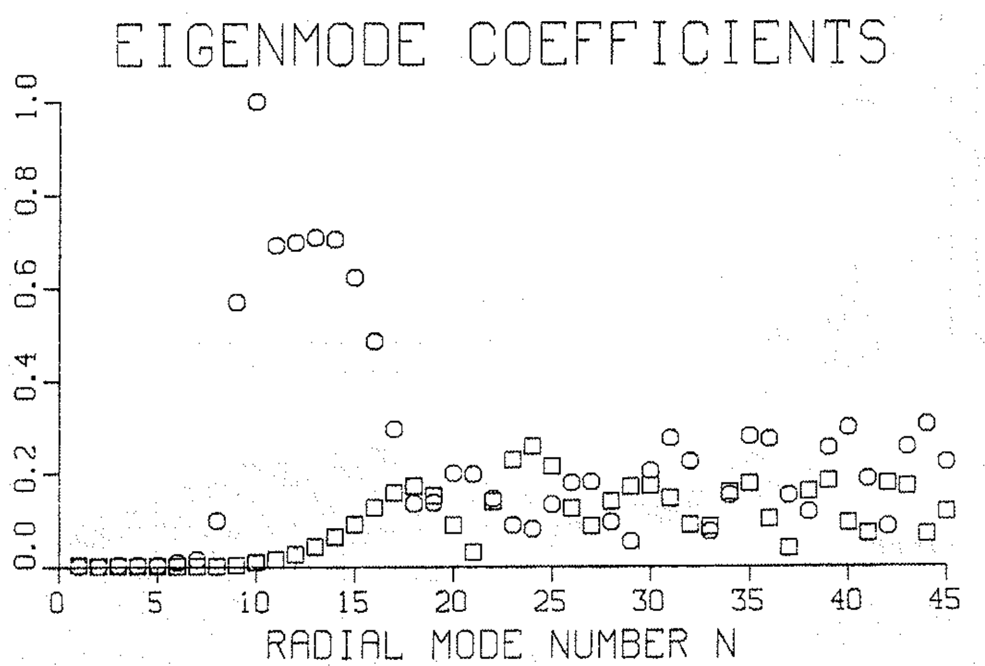
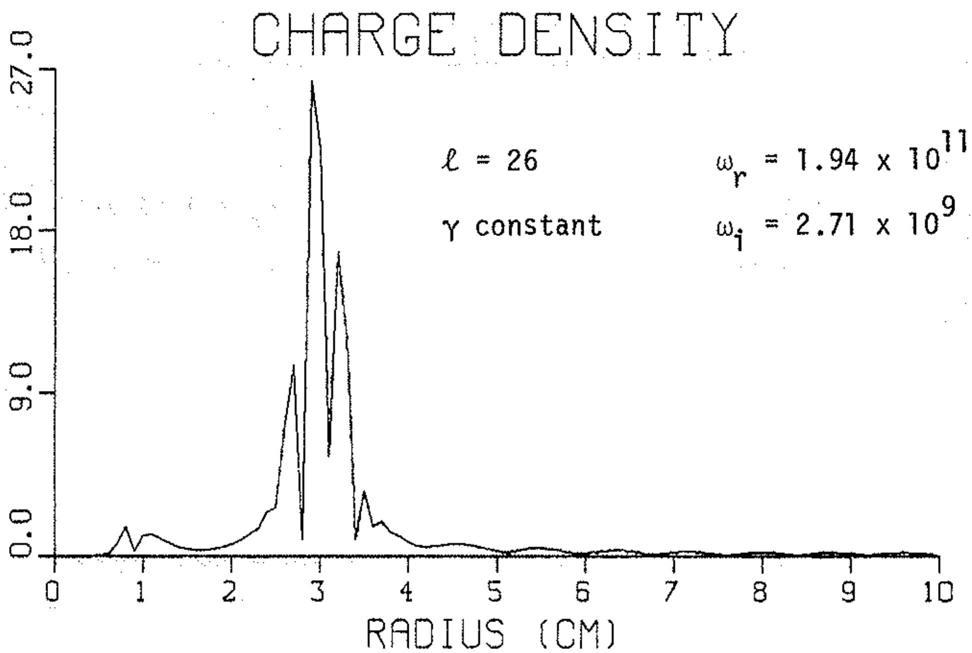

EIGENMODE COEFFICIENTS
0.0 0.2 0.4 0.6 0.8 1.0
0 5 10 15 20 25 30 35 40 45
RADIAL MODE NUMBER N

Fig. 3.71


CHARGE DENSITY
27.0
18.0
9.0
0.0
ℓ = 26
γ constant
ω_r = 1.94 x 10^{11}
ω_i = 2.71 x 10^9
0 1 2 3 4 5 6 7 8 9 10
RADIUS (CM)

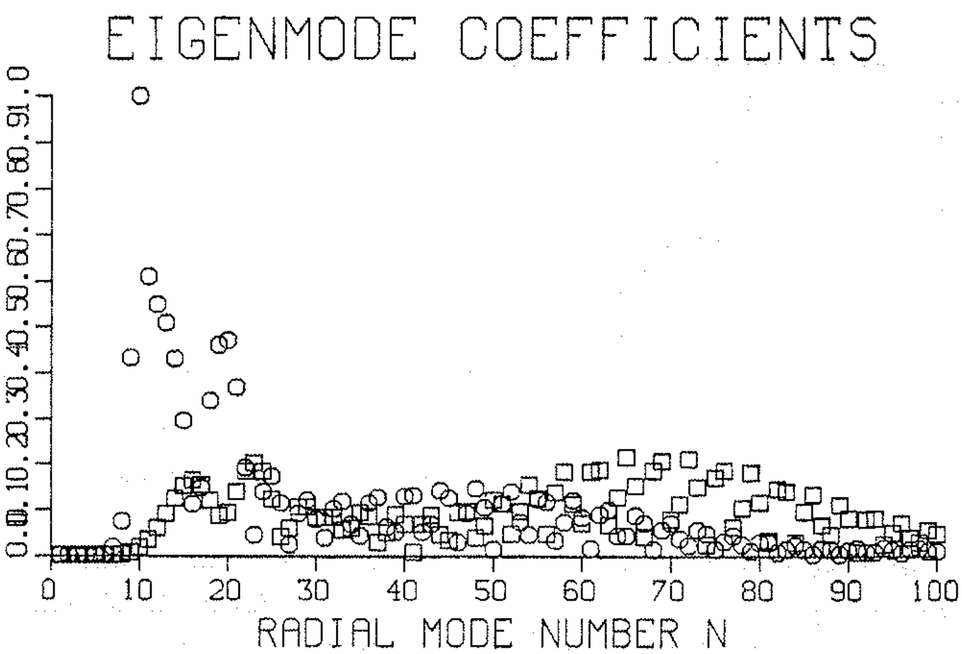
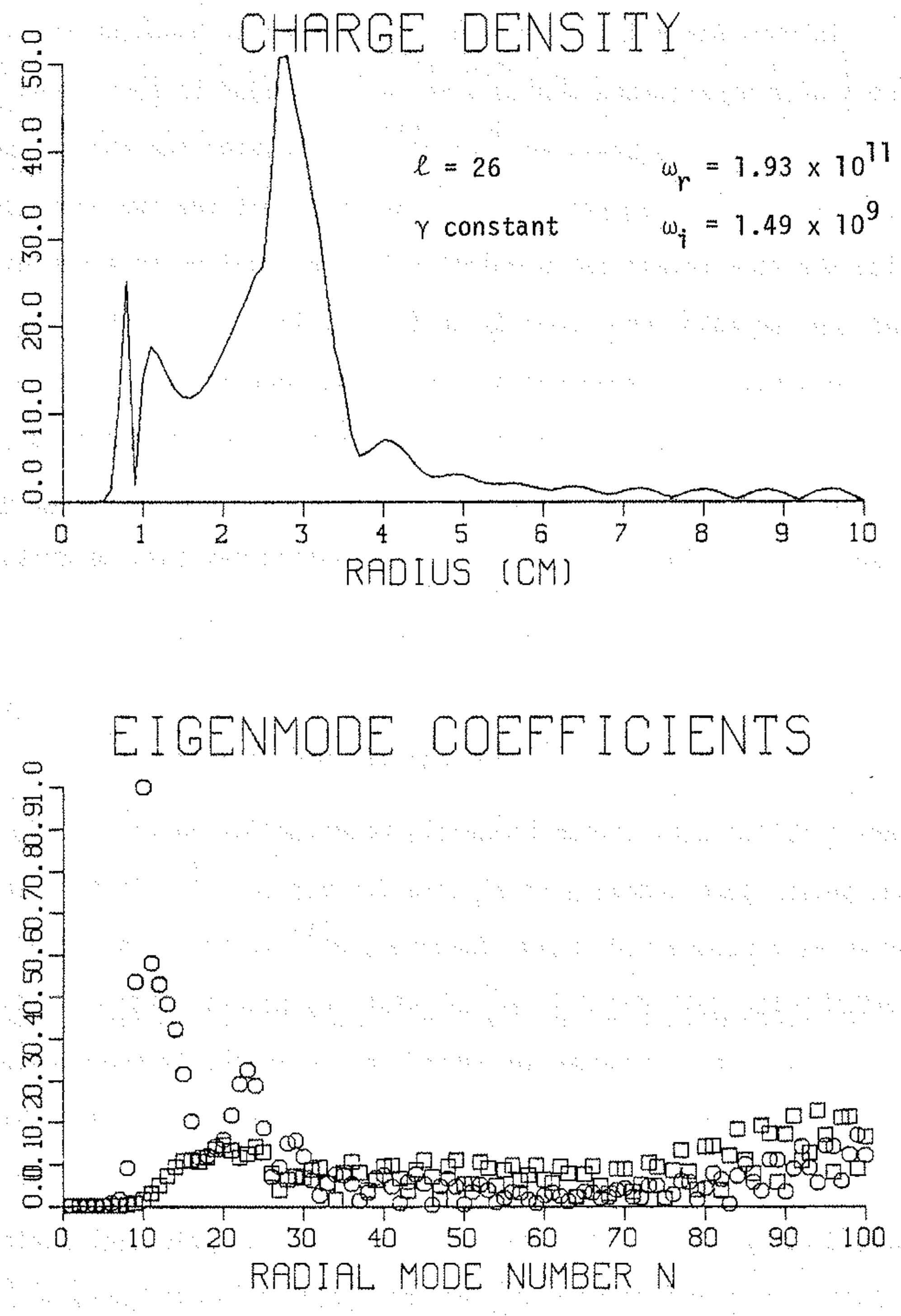

EIGENMODE COEFFICIENTS
0 10 20 30 40 50 60 70 80 90 100
RADIAL MODE NUMBER N

Fig. 3.72


CHARGE DENSITY
ℓ = 26
γ constant
$\omega_r = 1.93 \times 10^{11}$
$\omega_i = 1.49 \times 10^9$
0.0
10.0
20.0
30.0
40.0
50.0
0 1 2 3 4 5 6 7 8 9 10
RADIUS (CM)
EIGENMODE COEFFICIENTS
0 10 20 30 40 50 60 70 80 90 100
RADIAL MODE NUMBER N

APPENDIX A

Equivalence to Method of Lewis and Symon

In this Appendix we show that the method of handling the integrals over unperturbed orbits used in this paper is equivalent to a method proposed by Lewis and Symon[1] for solving the inhomogeneous linearized Vlasov equation. First we show that the two approaches give the same result for problems with one non-ignorable coordinate, and then we make some comments on the difficulties which arise in problems with more than one non-ignorable coordinate.

We denote the non-ignorable coordinates and momenta by $\underline{Q}$ and $\underline{P}$, respectively, and the ignorable coordinates and momenta by $\underline{q}$, $\underline{p}$, respectively. The linearized Vlasov equation may then be written:

$$\begin{aligned}(\frac{\partial}{\partial t} + L)\ f^{(1)}(\underline{Q},\underline{q},\underline{P},\underline{p},t) &= \frac{d}{dt}\ f^{(1)} \\ &= U(\underline{Q},\ \frac{\partial}{\partial \underline{q}},\ \frac{\partial}{\partial t},\ \underline{P},\underline{p})\ \phi^{(1)}(\underline{Q},\underline{q},t)\ .\end{aligned} \tag{A.1}$$

Here L is the equilibrium Liouville operator (as in (1.8)), $\frac{d}{dt}$ is the phase space comoving total time derivative, $f^{(1)}$ is the first order perturbed distribution function, $\phi^{(1)}$ is the first order perturbed potential, and U is an operator representing the effect of $\phi^{(1)}$ on f_o. U is assumed to depend on the ignorable coordinates and time only through the derivatives $\frac{\partial}{\partial \underline{q}}$, $\frac{\partial}{\partial t}$; examples of typical U operators are seen on the right sides of (1.8) and (2.1). The perturbed potential $\phi^{(1)}$ may represent a vector of potential fields, as in the electromagnetic case; we will deal with only one potential function explicitly here to avoid unnecessary proliferation of indices. For the same reason we consider only one particle species.

The linearized field equation may be written

$$F(\underline{Q}, \frac{\partial}{\partial \underline{q}}, \frac{\partial}{\partial t})\ \phi^{(1)}(\underline{Q},\underline{q},t) = \int d\underline{P}d\underline{p}J(\underline{Q},\underline{P},\underline{p})\ f^{(1)}(\underline{Q},\underline{q},\underline{P},\underline{p},t)\ . \qquad (A.2)$$

Here F is field operator (such as the Laplacian or D'Alembertian) depending on the ignorable coordinates and time only through the derivatives, and J is a weighting function (independent of $\underline{q}$ and t) which typically makes the right side of (A.2) represent the charge or current density, as in (2.2).

We may modify Eqs. (A.1) and (A.2) by introducing the Lewis-Holdren substitution.[1] Given an operator

$$R(\underline{Q}, \frac{\partial}{\partial \underline{q}}, \underline{P},\underline{p}, \frac{\partial}{\partial t})$$

we define

$$g = f^{(1)} - R\phi^{(1)} \qquad (A.3)$$

so that (A.1) and (A.2) become

$$(\frac{\partial}{\partial t} + L)g = [U-(\frac{\partial}{\partial t} + L)R]\ \phi^{(1)} \qquad (A.4)$$

$$[F - \int d\underline{P}d\underline{p}JR]\ \phi^{(1)} = \int d\underline{P}d\underline{p}Jg\ . \qquad (A.5)$$

These equations are of the same form as (A.1) and (A.2) with the replacements

$$U \to U - \left(\frac{\partial}{\partial t} + L\right)R \ ,$$

$$f^{(1)} \to g + R\phi^{(1)} \ ,$$

$$F \to F - \int d\underline{P}d\underline{p}\mathcal{J}R \ .$$

The motivation for this substitution is to choose R so that the eigenfunctions of the new field operator will more closely resemble the eigenmode of the system, making the dispersion matrix more nearly diagonal and the resulting dispersion relation easier to solve. Since the substitution does not affect the form of the equations, it will not affect the methods to be described here, and we will continue to work with the equations in the notation of (A.1) and (A.2), assuming that the substitution, if any, has already been carried out.

We begin by assuming that the perturbed quantities have a time dependence $e^{-i\omega t}$ with $\mathrm{Im}(\omega) > 0$, since we are interested in unstable modes. We also assume that the system is periodic in the ignorable coordinates, so that we may Fourier analyze in $\underline{q}$ space:

$$f^{(1)}(\underline{Q},\underline{q},\underline{P},\underline{p},t) = \sum_{\underline{\kappa}} f^{(1)}_{\underline{\kappa}}(\underline{Q},\underline{P},\underline{p})\ e^{i\underline{\kappa}\cdot\underline{q}-i\omega t} \ , \qquad (A.6)$$

$$\phi^{(1)}(\underline{Q},\underline{q},t) = \sum_{\underline{\kappa}} \phi^{(1)}_{\underline{\kappa}}(\underline{Q})\ e^{i\underline{\kappa}\cdot\underline{q}-i\omega t} \ . \qquad (A.7)$$

Here if each ignorable coordinate q_k has period Q_k, then

$$\kappa_k = \frac{2\pi \ell_k}{Q_k} ,$$

where ℓ_k is an integer.

The Vlasov equation then becomes

$$\frac{d}{dt}\left[\sum_{\underline{\kappa}} f^{(1)}_{\underline{\kappa}}(\underline{Q},\underline{P},\underline{p})\, e^{i\kappa\cdot \underline{q} - i\omega t}\right] = \sum_{\underline{\kappa}'} U(\underline{Q}, i\underline{\kappa}', -i\omega, \underline{P}, \underline{p})\, \phi^{(1)}_{\underline{\kappa}'}(\underline{Q})\, e^{i\underline{\kappa}'\cdot \underline{q} - i\omega t} , \tag{A.8}$$

which we may solve by integration over unperturbed orbits:

$$\sum_{\underline{\kappa}} f^{(1)}_{\underline{\kappa}}(\underline{Q},\underline{P},\underline{p})\, e^{i\underline{\kappa}\cdot \underline{q} - i\omega t} = \sum_{\underline{\kappa}'} \int_{-\infty}^{t} dt' U(\underline{Q}', i\underline{\kappa}', -i\omega, \underline{P}', \underline{p})\, \phi^{(1)}_{\underline{\kappa}'}(\underline{Q}')\, e^{i\underline{\kappa}'\cdot \underline{q}' - i\omega t'} , \tag{A.9}$$

where here $\underline{Q}' = \underline{Q}'(t')$ denotes the unperturbed orbit of a particle as a function of t', subject to the condition that at t' = t,

$$Q'(t'=t) = \underline{Q} \tag{A.10}$$

and similarly for $\underline{P}'$ and $\underline{q}'$.

We now isolate one Fourier component on the right of (A.9) by multiplying by $e^{-i\underline{\kappa}\cdot \underline{q} + i\omega t}$ and integrating over $\underline{q}$. Using the identity

$$\underline{\kappa}'\cdot \underline{q}' - \underline{\kappa}\cdot \underline{q} = (\underline{\kappa}' - \underline{\kappa})\cdot \underline{q} - \underline{\kappa}'\cdot(\underline{q} - \underline{q}')$$

and the fact that by virtue of (A.10) $\underline{q} - \underline{q}'$ is independent of $\underline{q}$, we have

$$f_{\underline{\kappa}}^{(1)}(\underline{Q},\underline{P},\underline{p}) = \int_{-\infty}^{t} dt'\ e^{i\omega(t-t')} U(\underline{Q}',i\underline{\kappa},-i\omega,\underline{P}',\underline{p})\ \phi_{\underline{\kappa}}^{(1)}(\underline{Q}')\ e^{i\underline{\kappa}\cdot(\underline{q}'-\underline{q})} . \tag{A.11}$$

Similarly, we may isolate one term in the field equation (A.2) and write

$$F(\underline{Q},i\underline{\kappa},i\omega)\ \phi_{\underline{\kappa}}^{(1)}(\underline{Q}) = \int d\underline{P}d\underline{p}\mathcal{J}(\underline{Q},\underline{P},p)\ f_{\underline{\kappa}}^{(1)}(\underline{Q},\underline{P},\underline{p}) . \tag{A.12}$$

The problem is now seen to be diagonal in $\underline{\kappa}$, so we drop the index $\underline{\kappa}$ from $f^{(1)}$, $\phi^{(1)}$, U, and F.

Next we expand the perturbed potential in eigenfunctions of the field operator:

$$\phi^{(1)}(\underline{Q}) = \sum_n \alpha_n \phi_n(\underline{Q}) , \tag{A.13}$$

where

$$F(\underline{Q},i\omega)\ \phi_n(\underline{Q}) = \lambda_n(\omega)\ \phi_n(\underline{Q})$$

and

$$\int d\underline{Q}\phi_{n'}^{*}(\underline{Q})\ \phi_n(\underline{Q}) = \delta_{nn'} .$$

Then, with the aid of (A.11), we may write (A.12) as

$$\sum_n \lambda_n(\omega)\, \alpha_n \phi_n(Q) = \int d\underline{P} d\underline{p} J(\underline{Q},\underline{P},\underline{p}) \int_{-\infty}^{t} dt'\, e^{i\omega(t-t')}$$

$$\cdot\, U(\underline{Q}',-i\omega,\underline{P}',\underline{p}') \sum_{n'} \alpha_{n'}\phi_{n'}(Q')\, e^{i\underline{\kappa}\cdot(\underline{q}'-\underline{q})} \; . \qquad \text{(A.14)}$$

Multiplying by $\phi_n^*(\underline{Q})$ and integrating over $\underline{q}$, we have

$$\lambda_n(\omega)\, \alpha_n = \sum_{n'} \int d\underline{Q} d\underline{P} d\underline{p} J(\underline{Q},\underline{P},\underline{p})\, \phi_n^*(\underline{Q})\, e^{-i\underline{\kappa}\cdot\underline{q}}$$

$$\cdot \int_{-\infty}^{t} dt'\, e^{i\omega(t-t')}\, U(\underline{Q}',-i\omega,\underline{P}',\underline{p}')\, \phi_{n'}(\underline{Q}')\, e^{i\underline{\kappa}\cdot\underline{q}'}\, \alpha_{n'} \; .$$

which we then write as

$$D_{nn'}(\omega)\, \alpha_{n'} = 0 \; ,$$

where the dispersion matrix D is given by

$$D_{nn'}(\omega) = \lambda_n(\omega)\, \delta_{nn'} - \int d\underline{Q} d\underline{P} d\underline{p} J(\underline{Q},\underline{P},\underline{p})\, \phi_n^*(\underline{Q})\, e^{-i\underline{\kappa}\cdot\underline{q}}$$

$$\cdot \int_{-\infty}^{t} dt'\, e^{i\omega(t-t')}\, U(\underline{Q}',-i\omega,\underline{P}',\underline{p})\, \phi_{n'}(\underline{Q}')\, e^{i\underline{\kappa}\cdot\underline{q}'} \; . \qquad \text{(A.15)}$$

Now we specialize to the case of one non-ignorable coordinate, $\underline{Q} = Q$. If the system is bounded and non-asymptotic, Q will be periodic in t with period $T(E,\underline{p})$, where E is the energy of the particle. Since $\dot{\underline{q}} = \frac{\partial H}{\partial p}$, $\dot{\underline{q}}$ will also be periodic with period T, so we can write

$$\underline{q} = \beta t + \tilde{\underline{q}}(t) \; ,$$

where $\tilde{\underline{q}}(t)$ is a periodic function of t. Similarly, assuming P is bounded, it must be periodic with period T. The function

$$U(Q,-i\omega,P,\underline{p})\ \phi_n(Q)\ e^{i\underline{\kappa}\cdot\tilde{\underline{q}}}$$

is then periodic in t, and we expand it in a Fourier series:

$$U(Q,-i\omega,P,p)\ \phi_n(Q)\ e^{i\underline{\kappa}\cdot\tilde{\underline{q}}} = \frac{1}{T^{1/2}} \sum_m \langle m|U|n\rangle\ e^{im\Omega t}\ , \qquad \text{(A.16)}$$

where $\Omega = \frac{2\pi}{T}$ and we have called the expansion coefficient $\langle m|U|n\rangle$ for reasons that will become apparent later. Now we may perform the time integral in (A.15):

$$\int_{-\infty}^{t} dt'\ e^{i\omega(t-t')}\ U(Q',-i\omega,P',\underline{p})\ \phi_{n'}(Q')\ e^{i\underline{\kappa}\cdot\underline{q}'}$$

$$= \sum_m \frac{\langle m|U|n'\rangle e^{i(m\Omega+\underline{\kappa}\cdot\underline{\beta})t}}{i(m\Omega+\underline{\kappa}\cdot\underline{\beta}-\omega)T^{1/2}}\ .$$

Similarly, the function

$$\mathcal{J}(Q,P,\underline{p})\ \phi_n^*(Q)\ e^{i\underline{\kappa}\cdot\tilde{\underline{q}}}$$

is periodic and may be expanded:

$$\mathcal{J}(Q,P,\underline{p})\ \phi_n^*(Q)\ e^{i\underline{\kappa}\cdot\tilde{\underline{q}}} = \frac{1}{T^{1/2}} \langle n|\mathcal{J}|m'\rangle\ e^{-im'\Omega t}\ ,$$

and the dispersion matrix (A.15) may then be written

$$D_{nn'}(\omega) = \lambda_n(\omega)\ \delta_{nn'} + \sum_{m,m'} \int dQdPd\underline{p}\ \frac{1}{T}\ \frac{i\langle n|\mathcal{J}|m'\rangle\langle m|U|n'\rangle e^{i(m-m')\Omega t}}{m\Omega+\underline{\kappa}\cdot\underline{\beta}-\omega}\ . \qquad \text{(A.17)}$$

We make the canonical transformation of integration variables $dQdP \rightarrow dtdH$, where H is the Hamiltonian, and perform the resulting time integral:

$$D_{nn'}(\omega) = \lambda_n(\omega)\ \delta_{nn'} + \sum_m \int dHd\underline{p}\ \frac{i\langle n|J|m\rangle\langle m|U|n'\rangle}{m\Omega+\underline{k}\cdot\underline{\beta}-\omega} \quad . \tag{A.18}$$

Now we show that the same result may be obtained by the method of Lewis and Symon,[1] which involves expanding the perturbed distribution function in eigenfunctions of the Liouville operator:

$$f^{(1)}(\underline{Q},\underline{q},\underline{P},\underline{p},t) = \sum_r \gamma_r w_r(\underline{Q},\underline{q},\underline{P},\underline{p})\ e^{-i\omega t} \ , \tag{A.19}$$

where w_r is the Liouville eigenfunction:

$$L w_r(\underline{Q},\underline{q},\underline{P},\underline{p}) = i\mu_r w_r(\underline{Q},\underline{q},\underline{P},\underline{p}) \ .$$

Here μ_r is real since L is anti-Hermitian.[1] Since $\frac{\partial}{\partial \underline{q}}$, $\underline{p}$, and H commute with the Liouville operator, we choose w_r to be an eigenfunction of all these operators:

$$w_r(\underline{Q},\underline{q},\underline{P},\underline{p}) = \overline{w}_r(\underline{Q},\underline{P},\underline{p})\ e^{i\underline{\kappa}\cdot\underline{q}}\ \delta(H-E)\ \delta(\underline{p}-\underline{p}_r) \ .$$

The index r therefore represents the vector $\underline{\kappa}$, the constants of the motion E and $\underline{p}_r$, and any other necessary indices. Thus the sum in (A.19) is actually a generalized sum, at least part of which is a continuous integral, but we continue to write it as a sum for convenience. The eigenfunctions are normalized so that

$$\int dQdqdPdp w^*_{r'}(Q,q,P,p)\ w_r(Q,q,P,p) = \delta_{rr'} \ .$$

Using (A.19), the Vlasov equation (A.1) may be written:

$$\sum_r i(\mu_r-\omega)\ \gamma_r w_r(Q,q,P,p)\ e^{-i\omega t}$$

$$= \sum_{\kappa'} U(Q,i\kappa',-i\omega,P,p)\ \phi^{(1)}_{\kappa'}(Q)\ e^{i\kappa'\cdot q-i\omega t} \ .$$

Multiplying by w^*_r on both sides and integrating over all phase space, we have

$$i(\mu_r-\omega)\ \gamma_r = \int dQdP\overline{w}^*_r(Q,P,p_r)\ U(Q,i\kappa,-i\omega,P,p_r)\ \phi^{(1)}_{\kappa}(Q)\ \delta(H-E) \ . \tag{A.20}$$

The problem is again seen to be diagonal in κ (remember r includes κ) so we drop the index κ. Using the expansion (A.13), (A.20) becomes

$$i(\mu_r-\omega)\ \gamma_r = \sum_{n'} \int dQdP\overline{w}^*_r(Q,P,p_r)\ U(Q,-i\omega,P,p_r)\ \phi_{n'}(Q)\ \delta(H-E)\ \alpha_{n'}$$

or

$$\gamma_r = \sum_{n'} \frac{\langle r|U|n'\rangle}{i(\mu_r-\omega)}\ \alpha_{n'} \ , \tag{A.21}$$

where we have defined

$$\langle r|U|n\rangle = \int dQdqdPdp w^*_r(Q,q,P,p)\ U(Q,i\kappa,-i\omega,P,p)\ \phi_n(Q)\ e^{i\kappa\cdot q}$$

$$= \int dQdP\overline{w}^*_r(Q,P,p_r)\ U(Q,-i\omega,P,p_r)\ \phi_n(Q)\ \delta(H-E) \ . \tag{A.22}$$

The field equation (A.12) becomes

$$\sum_n \lambda_n(\omega)\ \alpha_n \phi_n(\underline{Q}) = \sum_r \int d\underline{P}\mathcal{J}(\underline{Q},\underline{P},\underline{p}_r)\ \bar{w}_r(\underline{Q},\underline{P},\underline{p}_r)\ \delta(H-E)\ \gamma_r \ .$$

Multiplying by $\phi_n^*(\underline{Q})$, integrating over $\underline{Q}$, and using (A.20), we have

$$\lambda_n(\omega)\ \alpha_n = \sum_{r,n'} \frac{<n|\mathcal{J}|r><r|U|n'>}{i(\mu_r-\omega)}\ \alpha_{n'} \ , \qquad \text{(A.23)}$$

where we have defined

$$<n|\mathcal{J}|r> = \int dQd\underline{P}\phi_n^*(\underline{Q})\ \mathcal{J}(\underline{Q},\underline{P},\underline{p}_r)\ \bar{w}_r(\underline{Q},\underline{P},\underline{p}_r)\ \delta(H-E)\ .$$

We may write (A.23) as

$$D_{nn'}(\omega)\ \alpha_{n'} = 0 \ ,$$

where

$$D_{nn'}(\omega) = \lambda_n(\omega)\ \delta_{nn'} + \sum_r \frac{i<n|\mathcal{J}|r><r|U|n'>}{(\mu_r-\omega)} \ . \qquad \text{(A.24)}$$

It is shown in Ref. (1) that the Liouville eigenfunction for the one non-ignorable coordinate case may be written in the form

$$\bar{w}_r = \frac{1}{T^{1/2}}\ e^{-i\underline{\kappa}\cdot\underline{q}(\tau)}\ e^{im\Omega\tau} \ .$$

Here we have converted to our notation; m is an integer and τ is a parameter representing time along an unperturbed trajectory. Substituting this expression into (A.22), and converting $\int dQdP$ to $\int dHdt$, we obtain

$$<r|U|n> = \frac{1}{T^{1/2}} \int_0^{\tau} d\tau \; e^{-im\tau} \; \phi_n(Q) \; U(Q,-i\omega,P,\underline{p}_r) \; e^{i\underline{\kappa}\cdot\underline{q}(\tau)}$$

so that $<r|U|n>$ is exactly the Fourier coefficient appearing in (A.16). A similar result is easily obtained relating the two definitions of $<n|J|r>$. It is also shown in Ref. (1) that for the one non-ignorable coordinate case,

$$\mu_r = m\Omega + \underline{\kappa} \cdot \underline{\beta} \; ,$$

where m is an integer indexing the functions of the Liouville operator. Thus taking $\sum_r$ as $\sum_m \int dHdp_r$, we have exactly the result (A.18), demonstrating that integration over unperturbed orbits and expansion in Liouville eigenfunctions lead to equivalent representations of the dispersion matrix. Of course, we knew that a priori, since both methods solve the same problem; the benefit of showing it explicitly is the insight it affords into how each method works.

For systems with more than one non-ignorable coordinate, the Liouville eigenfunctions will in general be discontinous everywhere.(1) Thus the integrals represented by the inner products in (A.24) must be taken as Lebesque integrals, and even so their existence is questionable. In terms of the integration over unperturbed orbits, the problem may be stated as follows: can each coordinate Q of the system be represented as a function of time in the following form:

$$Q(t) = \sum_k a_k e^{i\omega_k t} , \tag{A.25}$$

where the coefficients a_k and the frequencies ω_k depend on the initial conditions for the orbit. Such a function has been called "almost periodic" by Bohr,[(18)] who showed that such a series possesses many of the properties of a Fourier series. If it has a finite number of terms, it is called "quasi-periodic."

The expression of the motion of a particle in a series of the form (A.25) is one of the most important techniques of celestial mechanics,[(19)] and a method for developing such series for an arbitrary Hamiltonian system has been presented by Whittaker.[20] The question then is whether or not the series converges, and this is a well-known and still unsolved problem in celestial mechanics. The difficulty is that the coefficients a_k in (A.25) turn out to have terms which are linear combinations of the frequencies such as

$$(n_1\omega_1 + n_2\omega_2 + \ldots + n_N\omega_N) \tag{A.26}$$

in their denominators, where the n's are integers. Even if the ω_k's are rationally independent, sets of n's can be found which make (A.26) arbitrarily small. The larger the order of the term in (A.25), the larger and more numerous will be the integers in the combination (A.26), and so the smaller the denominator can be. But also (hopefully), the numerators of these terms will become smaller with increasing order. The convergence of the series then depends on how fast the numerators decrease with the increasing order of approximation as more numerous and larger integers enter the combinations (A.26). This is the famous problem of the "small denominators."

Whittaker's method of obtaining such series depends on constructing additional constants of the motion (which he calls "adelphic integrals") in series form. The question of convergence of (A.25) may also be considered in terms of the convergence of this series, or in other words, the question of whether there exist in general certain constants of the motion other than the energy. Poincare[21] has shown that if such integrals exist, they cannot be analytic in their dependence on the initial conditions. Thus the series (A.25) cannot converge uniformly in time on one hand, and on the other hand, for all values of the initial conditions within certain limits. Whittaker's adelphic integral series, however, do not depend analytically on the initial conditions; in fact, they change in form whenever the ratio of two parameters in the initial conditions changes from rational to irrational or vice-versa. Whittaker felt strongly that the adelphic integrals did exist in general, but was able to shown this only in special cases. Thus the series are widely regarded as merely formal results, though to quote Giacaglia:[22] "earlier results by Poincare were considered in a much too general form and thought to prevent the slightest chance of integrability of dynamical systems. The only thing one can conjecture is that non-integrable systems are dense, say in the space of all Hamiltonian functions. No statement is however available on the density of integrable systems. If they are at least as dense as the rational numbers on any segment, we might still say there are quite a few integrable systems." And to quote Brouwer:[19] "These comments serve to emphasize that the problem of the small denominators is a basic feature of all of celestial mechanics. Its mathematical nature is still imcompletely understood."

One result which is available for Hamiltonians with two degrees of freedom (under certain mild restrictions) is a theorem due to Kolmogorov and Arnold,[26] which asserts that in a neighborhood of an integrable system the quasi-periodic solutions have positive measure, and in a sense are a majority.[27] However, this theorem does not extend to more than two degrees of freedom.

One might wonder why we need be concerned with long-time behavior of the expansion, since in plasma physics we are concerned only with short-time behavior. The answer is that we use a time integral (as in going from (A.16) to (A.17)) to do part of the integration over phase space. Thus if our expression for the particle motion is inaccurate for large t, the integral will be inaccurate on some regions of phase space. Another difficulty is that if the series (A.23) does not converge uniformly in time, the series in the dispersion matrix elements corresponding to the sum over m in (A.17) will not converge.

If, however, the Hamiltonian for a problem is "close enough" to an integrable Hamiltonian (as it will certainly be if integrable systems are dense), we may apply the methods of this section to the integrable Hamiltonian, obtaining convergent series results. These we expect will be "close" to the actual behavior of the plasma system, since we do not expect that a slight change in the Hamiltonian will make a large change in the properties of the plasma. This indeed is a fundamental assumption of any analysis, since any Hamiltonian we write down can be only an approximation to the true Hamiltonian.

To summarize: if we can find an integrable Hamiltonian which approximates the actual Hamiltonian for the problem, we can express the particle trajectories as quasi-periodic functions of time and apply the method of this Appendix to the stability analysis. The practicality of this approach, of course, depends on how many terms of the series (A.25) must be retained, and this must be determined for each case individually.

APPENDIX B

Some Bessel Function Formulae

In this Appendix are collected some derivations of results used earlier. First we wish to derive the expansions of the divergence and curl of the vector potential in terms of cylindrical harmonics that were used in Chapter II.

We evaluate the divergence first. From (2.14) we have

$$\frac{\partial A_z}{\partial z} = \sum_{\ell,n,k_z} ik_z \, \gamma_{n\ell k} \, J_\ell(\lambda_{n\ell} r) \, e^{i(\ell\theta + k_z z)} \quad .$$

It remains to evaluate

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = \nabla_\perp \cdot \underline{A}_\perp \; .$$

It is easiest to transform to polar coordinates using

$$x = r\cos\theta \quad , \quad y = r\sin\theta \; .$$

Then

$$\nabla_\perp \cdot \underline{A} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial\theta}\right) \frac{A^+ + A^-}{2} + \left(\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial\theta}\right) \frac{A^+ - A^-}{2i}$$

$$= \frac{1}{2} e^{-i\theta} \frac{\partial}{\partial r} A^+ - \frac{i}{2} \frac{e^{-i\theta}}{r} \frac{\partial}{\partial\theta} A^+ + \frac{1}{2} e^{i\theta} \frac{\partial}{\partial r} A^- + \frac{i}{2} \frac{e^{i\theta}}{r} \frac{\partial}{\partial\theta} A^-$$

$$= \frac{1}{2} \frac{\partial}{\partial r} [e^{-i\theta} A^+ + e^{i\theta} A^-] - \frac{i}{2r} [e^{-i\theta} \frac{\partial}{\partial\theta} A^+ - e^{i\theta} \frac{\partial}{\partial\theta} A^-] \; .$$

Using (2.14) we then expand A^+ and A^- in cylindrical harmonics and get:

$$e^{-i\theta}A^+ + e^{i\theta}A^- = \sum_{n,\ell,k} [\alpha_{n\ell k}\, J_{\ell+1}(\lambda'_{n\ell}r) + \beta_{n\ell k}\, J_{\ell+1}(\lambda_{n\ell}r)$$

$$+ \alpha_{n\ell k}\, J_{\ell-1}(\lambda'_{n\ell}r) - \beta_{n\ell k}\, J_{\ell-1}(\lambda_{n\ell}r)]\, e^{i(\ell\theta+k_z z)},$$

$$\frac{1}{2}\frac{\partial}{\partial r}(e^{-i\theta}A^+ + e^{i\theta}A^-) = \frac{1}{2}\sum_{n,\ell,k}\{\alpha_{n\ell k}\lambda'_{n\ell}\, [J'_{\ell+1}(\lambda'_{n\ell}r) + J'_{\ell-1}(\lambda'_{n\ell}r)]$$

$$+ \beta_{n\ell k}\lambda_{n\ell}\, [J'_{\ell+1}(\lambda_{n\ell}r) - J'_{\ell-1}(\lambda_{n\ell}r)]\}\, e^{i(\ell\theta+k_z z)}$$

and

$$e^{-i\theta}\frac{\partial}{\partial\theta}A^+ - e^{i\theta}\frac{\partial}{\partial\theta}A^- = i\sum_{\ell,n,k}\{\alpha_{n\ell k}[(\ell+1)J_{\ell+1}(\lambda'_{n\ell}r)-(\ell-1)J_{\ell-1}(\lambda'_{n\ell}r)]$$

$$+ \beta_{n\ell k}[(\ell+1)J_{\ell+1}(\lambda_{n\ell}r)+(\ell-1)J_{\ell-1}(\lambda_{n\ell}r)]\}\, e^{i(\ell\theta+k_z z)}$$

Noting that the $\alpha_{n\ell}$'s occur in terms with Bessel functions of argument $\lambda'_{n\ell}r$ and the $\beta_{n\ell}$'s in terms with Bessel functions of $\lambda_{n\ell}r$, we drop the arguments of the Bessel functions to simplify notation and write

$$-\frac{i}{2r}[e^{-i\theta}\frac{\partial}{\partial\theta}A^+ - e^{i\theta}\frac{\partial}{\partial\theta}A^-] = \sum_{\ell,n,k}\{\alpha_{n\ell k}[\frac{\ell}{2r}(J_{\ell+1}-J_{\ell-1}) + \frac{1}{2r}(J_{\ell+1}+J_{\ell-1})]$$

$$+ \beta_{n\ell k}[\frac{\ell}{2r}(J_{\ell+1}+J_{\ell-1}) + \frac{1}{2r}(J_{\ell+1}-J_{\ell-1})\}\, e^{i(\ell\theta+k_z z)}$$

and

$$\frac{1}{2}\frac{\partial}{\partial r}[e^{-i\theta}A^{+} + e^{i\theta}A^{-}] = \sum_{\ell,n}\{\alpha_{n\ell k}[\frac{\lambda'_{n\ell}}{2}(J'_{\ell+1}+J'_{\ell-1})]$$

$$+ \beta_{n\ell k}[\frac{\lambda_{n\ell}}{2}(J'_{\ell+1}-J'_{\ell-1})]\} e^{i(\ell\theta+k_z z)}$$

Combining results,

$$\nabla_{\perp} \cdot \underline{A}_{\perp} = \sum_{\ell,n,k}\{\alpha_{n\ell k}[\frac{\ell}{2r}(J_{\ell+1}-J_{\ell-1}) + \frac{1}{2r}(J_{\ell+1}+J_{\ell-1}) + \frac{\lambda'_{n\ell}}{2}(J'_{\ell+1}+J'_{\ell-1})]$$

$$+ \beta_{n\ell k}[\frac{\ell}{2r}(J_{\ell+1}+J_{\ell-1}) + \frac{1}{2r}(J_{\ell+1}-J_{\ell-1}) + \frac{\lambda_{n\ell}}{2}(J'_{\ell+1}-J'_{\ell-1})]\}$$

$$\cdot e^{i(\ell\theta+k_z z)} .$$

Using the Bessel function identities

$$\frac{\ell}{x}J_{\ell}(x) + J'_{\ell}(x) = J_{\ell-1}(x)$$

$$\frac{\ell}{x}J_{\ell}(x) - J'_{\ell}(x) = J_{\ell+1}(x) \qquad (B.3)$$

it is straightforward to show that (B.2) reduces to

$$\nabla_{\perp} \cdot \underline{A} = \sum_{\ell,n,k}\beta_{n\ell k}\lambda_{n\ell}J_{\ell}(\lambda_{n\ell}r)e^{i(\ell\theta+k_z z)} \qquad (B.4)$$

and combining (B.2) and (B.3) we obtain finally

$$\nabla \cdot \underline{A} = \sum_{\ell,n,k}[\beta_{n\ell k}\lambda_{n\ell} + ik_z\gamma_{n\ell k}]J_{\ell}(\lambda_{n\ell}r)e^{i(\ell\theta+k_z z)} . \qquad (B.5)$$

Next we calculate the curl. Since in the electromagnetic calculations we have taken $\frac{\partial}{\partial z} = A_z = 0$, we need only calculate the z-component with $k_z = 0$:

$$(\nabla\times\underline{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial\theta}) \frac{A^+-A^-}{2i}$$

$$- (\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos \frac{\partial}{\partial\theta}) \frac{A^++A^-}{2}$$

$$= \frac{i}{2}\frac{\partial}{\partial r}(e^{i\theta}A^- - e^{-i\theta}A^+) - \frac{1}{2}(e^{-i\theta}\frac{1}{r}\frac{\partial}{\partial\theta}A^+ + e^{i\theta}\frac{1}{r}\frac{\partial}{\partial\theta}A^-) .$$

Using (2.14) and suppressing the arguments of the Bessel functions as above, we have

$$\frac{i}{2}\frac{\partial}{\partial r}(e^{i\theta}A^- - e^{-i\theta}A^+) =$$

$$\frac{i}{2}\sum_{\ell,n}[\alpha_{n\ell}\lambda'_{n\ell}(J'_{\ell-1}-J'_{\ell+1}) - \beta_{n\ell}\,\lambda_{n\ell}(J'_{\ell+1}+J'_{\ell-1})]\,e^{i\ell\theta}$$

$$-\frac{1}{2}(e^{-i\theta}\frac{1}{r}\frac{\partial}{\partial\theta}A^+ + e^{i\theta}\frac{1}{r}\frac{\partial}{\partial\theta}A^-) = -\frac{i}{2}[\frac{\ell+1}{r}e^{-i\theta}A^+ + \frac{\ell-1}{r}e^{i\theta}A^-]$$

$$= -\frac{i}{2}\sum_{\ell,n}(\frac{\ell+1}{r}\alpha_{n\ell}J_{\ell+1} + \frac{\ell+1}{r}\beta_{n\ell}J_{\ell+1} + \frac{\ell-1}{r}\alpha_{n\ell}J_{\ell-1}$$

$$-\frac{\ell-1}{r}\beta_{n\ell}J_{\ell-1})\,e^{i\ell\theta} .$$

Combining these results we have

$$(\nabla\times\underline{A})_z = \frac{i}{2}\sum_{\ell,n}[\alpha_{n\ell}\lambda'_{n\ell}(J'_{\ell-1}-J'_{\ell+1}) - \beta_{n\ell}\lambda_n(J'_{\ell+1}+J'_{\ell-1}) - \frac{\ell+1}{r}\alpha_{n\ell}J_{\ell+1}$$

$$-\frac{\ell+1}{r}\beta_{n\ell}J_{\ell+1} - \frac{\ell-1}{r}\alpha_{n\ell}J_{\ell-1} + \frac{\ell+1}{r}\beta_{n\ell}J_{\ell-1}]\,e^{i\ell\theta}$$

and using the identities (B.3) it may be shown that this expression reduces to

$$(\nabla\times\underline{A})_z = -i \sum_{\ell,n} \alpha_{n\ell}\, \lambda'_{n\ell}\, J_\ell(\lambda'_n\, r)\, e^{i\ell\theta} \; .$$

Next we wish to establish the two integral identities (3.50) and (3.51). First we note that if $f(x)$ is a function which has derivatives of all orders in a neighborhood of $x = 0$ and $f(0) = 0$, then

$$\int_o^R dx\, \frac{1}{x}\, \delta(x)\; f(x) = \frac{1}{2}\, f'(0) \tag{B.6}$$

which follows directly on expanding $f(x)$ in a Maclaurin series. We also note that the only non-vanishing Bessel functions and low-order derivatives at zero are:

$$\begin{aligned} J_o(0) &= 1 \quad , \quad J'_1(0) = \frac{1}{2} = -J'_{-1}(0) \\ J''_o(0) &= \frac{1}{2} \quad , \quad J''_2(0) = \frac{1}{4} = J''_{-2}(0) \; . \end{aligned} \tag{B.7}$$

From (B.7) we easily see that

$$\frac{\partial}{\partial b}\left[bJ_{\ell+m}(\lambda'_{n\ell}b)\; J'_{\ell+m}(\lambda'_{n'\ell}b)\right]\Big|_{b=0} = 0$$

$$\frac{\partial}{\partial b}\left[J_{\ell+m}(\lambda'_{n\ell}b)\; J_{\ell+m}(\lambda'_{n'\ell}b)\right]\Big|_{b=0} = 0$$

for all ℓ, m. Thus we may use (B.6) to evaluate the integrals in

(3.50) and (3.51). We have

$$\int_o^R db\, \frac{1}{b}\, \delta(b)\, \frac{\partial}{\partial b}\, [bJ_{\ell+m}(\lambda'_{n\ell}b)\; J'_{\ell+m}(\lambda'_{n'\ell}b)]$$

$$= \frac{1}{2}\, [2\lambda'_{n\ell}J'_{\ell+m}(\lambda'_{n\ell}b)J'_{\ell+m}(\lambda'_{n\ell}b) + 2\lambda'_{n\ell}J'_{\ell+m}(\lambda'_{n\ell}b)J'_{\ell+m}(\lambda'_{n'\ell}b)]_{b=0}$$

$$= \frac{\lambda'_{n\ell}}{4}\, (\delta_{m,-\ell+1} + \delta_{m,-\ell-1}) - \frac{\lambda'_{n'\ell}}{2}\, \delta_{m,-\ell}$$

and

$$\int_o^R db\, \frac{1}{b}\, \delta(b)\, \frac{\partial}{\partial b}\, [J_{\ell+m}(\lambda'_{n\ell}b)\; J_{\ell+m}(\lambda'_{n'\ell}b)]$$

$$= \frac{1}{2}\, [\lambda'^{2}_{n\ell}\; J''_{\ell+m}(\lambda'_{n\ell}b) + 2\lambda'_{n\ell}\lambda'_{n'\ell}\; J_{\ell+m}(\lambda'_{n\ell}b)\; J'_{\ell+m}(\lambda'_{n'\ell}b)$$

$$+ \lambda'^{2}_{n'\ell}\; J_{\ell+m}(\lambda'_{n\ell}b)\; J''_{\ell+m}(\lambda'_{n'\ell}b)]_{b=0}$$

$$= \frac{1}{4}\, \lambda'_{n\ell}\lambda'_{n'\ell}(\delta_{m,-\ell+1} + \delta_{m,-\ell-1}) - \frac{1}{4}\, (\lambda'^{2}_{n\ell} + \lambda^{2}_{n'\ell})\, \delta_{m,-\ell}$$

which establishes (3.50) and (3.51).

REFERENCES

1. H. R. Lewis and K. R. Symon, J. Math. Phys. 20, 413 (1979).
2. J. D. Callen and G. E. Guest, Nucl. Fusion 13, 87 (1973).
3. C. N. Lashmore-Davies, Plasma Physics 11, 271 (1969).
4. D. Montgomery, Phys. Fluids 13, 1401 (1970).
5. G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, 1944), p. 394.
6. M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nuclear Fusion 1962 Supplement, Part One, p. 143.
7. R. C. Davidson, Phys. Fluids 19, 1189 (1976).
8. J. D. Jackson, Classical Electrodynamics (John Wiley and Sons, 1962), p. 181.
9. N. C. Christofilos and T. K. Fowler (1968), Rep. UCRL 50355.
10. W. W. Destler, D. W. Hudgings, M. J. Rhee, S. Kawasaki, and V. L. Granatstein, J. App. Phys. 48, 3291 (1977).
11. V. L. Granatstein, S. P. Schlesinger, M. Herndon, R. K. Parker, and J. A. Pasour, App. Phys. Lett. 30, 384 (1977).
12. C. D. Striffler and T. Kammash, Plasma Phys. 15, 729 (1973).
13. T. J. Fessenden and B. W. Stallard, CTR Annual Report UCRL-50002-70, pp. 3-27.
14. N. A. Krall and A. W. Trivelpiece, Principles of Plasma Physics (McGraw-Hill, 1973), p. 409.

15. C. E. Pearson, Handbook of Applied Mathematics (Van Nostrand, 1974), p. 255.

16. V. Granatstein, C. Roberson, G. Benford, D. Tzach and S. Robertson, Appl. Phys. Lett. 32, 88 (1978).

17. R. Buschauer and G. Benford, Monthly Notices Roy. Astron. Soc., 176, 335 (1976).

18. H. Bohr, Almost Periodic Functions (Chelsea, 1947).

19. D. Brouwer and G. Clemence, Methods of Celestial Mechanics (Academic Press, 1961), p. 298.

20. E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Dover, 1944), pp. 423-450.

21. H. Poincare, Les Méthodes Nouvelles de la Méchanique Célèste, Vol. 2 (Dover, 1957).

22. G. E. O. Giacaglia, Perturbation Methods in Non-Linear Systems (Springer-Verlag, 1972), p. 156.

23. N. A. Krall, in Advances in Plasma Physics (edited by A. Simon and W. B. Thompson), (Interscience, 1968), p. 162.

24. V. I. Arnold and A. Avez, Ergodic Problems of Classical Mechanics (Benjamin, 1968), pp. 95-96.

25. C. L. Siegel and J. K. Moser, Lectures on Celestial Mechanics (Springer-Verlag, 1971), p. 269.