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The role of electron density in magnetic turbulence

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Inertial range energy transfer, decorrelation, and energy spectra are studied analytically and numerically for strongly anisotropic magnetohydrodynamic (MHD) turbulence augmented by electron density evolution. The model is relevant to interstellar turbulence and magnetic turbulence in fusion devices. For long wavelengths (compared to the ion gyroradius), magnetic and kinetic energies are equipartitioned through interactions that decorrelate on the Alfvénic time scale. Internal energy transfer is governed by advection and decorrelates on the eddy turnover time scale. For short wavelengths, the roles of internal and kinetic energy reverse. Magnetic and internal energies are equipartitioned by the kinetic Alfvén interaction, while kinetic energy evolves under a decoupled fluid straining interaction. The spectral indices for magnetic, kinetic, and internal energies are $-3/2$, $-3/2$, and $-7/4$ for long wavelengths, and -2, $-5/3$, and -2 for short wavelengths. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1362531]

I. INTRODUCTION

In long wavelength hydromagnetic turbulence, electron density fluctuations are advected passively, affecting neither magnetic field nor flow. At shorter wavelengths, approaching and extending beyond the scale of the ion gyroradius, the electron density becomes dynamically active.[1] It feeds back on magnetic fluctuations through the electron pressure, while direct forcing of density fluctuations by the magnetic field through parallel compression of field-aligned current becomes stronger than the advection of electron density by the flow. In the spectrum of electron density fluctuations in the warm, diffuse component of the local interstellar medium (ISM) these effects are confined to at best two decades near the inner scale.[1] This represents only the very smallest observed scales of a spectrum that encompasses over 12 decades.[2] Notwithstanding, recent measurements of angular broadening of extragalactic radio sources indicates that at these scales the spectrum fall off steepens, consistent with the presence of new effects.[3] Short wavelength magnetoturbulence is relevant to the high frequency magnetic spectrum measured in the Madison Symmetric Torus (MST).[4] The spectrum also represents scales that are somewhat larger than the ion gyroradius, and, like the ISM spectrum, it decays at a rate that is steeper than the $-3/2$ power of magnetohydrodynamics (MHD).[5] Moreover, recent measurements suggest that electron density fluctuations are characterized not by an advective response, but by a response that is compressional.[6]

We examine these effects by considering a model for hydromagnetic turbulence that incorporates density evolution and is valid in long and short wavelength regimes. Though not essential to the physics considered here, the model includes drifts associated with mean gradients in current and density and is therefore referred to as the drift-Alfvén model. We consider inertial range dynamics in spectra driven at the largest scale of the system. The role of electron density is probed by choosing the driving scale and the smaller scales of the spectrum to lie in long or short wavelength limits, or to span both. Prior studies[1,7] have been limited to the long wavelength regime, and established that the density becomes increasingly passive as scales become larger. The counter propagation of smaller scale Alfvénic disturbances along larger scale fields (Alfvén effect)[8] was found to govern the decorrelation between fluctuations of the magnetic field and flow, *even when the turbulence is maximally anisotropic*, i.e., when the wave number along the mean magnetic field goes to zero. This is because the turbulent magnetic field, as a multiscaled fluctuation, has smaller scale components that do not align with the mean field (which is dominated by the largest scales), or even with the local magnetic field in some sizable subregion of the full domain. (The latter is dominated by the largest scales in the subregion. The field on a smaller scale points in some other direction.) While the propagation of Alfvén waves along the mean or local magnetic field can be suppressed by the vanishing of the wave number in either of those directions, perpendicular motions support the propagation of Alfvénic disturbances along the Fourier components of the field at smaller scale. All such disturbances contribute to the Alfvén effect.

The situation is illustrated in Fig. 1, which shows a field line that on scales less than a wavelength of the undulation differs locally from the mean field direction, which is vertical in the figure. A fluctuation anisotropy that eliminates Alfvénic propagation with respect to the local field can be represented as phase fronts parallel to the local field. On a smaller scale than that of the local field, the field points in other directions, represented as an undulation superimposed on the local field. This field crosses phase fronts and thus supports Alfvénic motion on yet a smaller scale. The figure at best represents 2 decades of spatial variation. In a spectrum like that of interstellar turbulence, with spatial variation

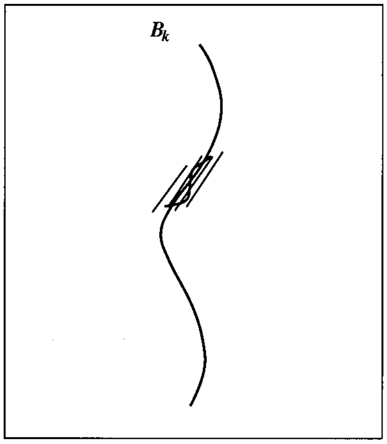

FIG. 1. Representation of a multiscaled turbulent magnetic field line with fluctuation phase fronts that have zero wave number along the local field. At smaller scale the field crosses these phase fronts. Fluctuations on yet a smaller scale can propagate along this field as Alfvén waves.

over 12 decades, these arguments can be applied in fractal fashion to ever smaller scales making it impossible to escape Alfvénic decorrelation. Fluctuations deep in the inertial range can decorrelate faster via Alfvénic propagation along field components whose scales are somewhat smaller than the scales of the mean or local field than they can from eddy straining, even though the field supporting the Alfvénic disturbance is weaker than the mean or local field. These statements are valid with respect to either the mean or local field, provided the turbulence is space filling and broadband, i.e., the field is multiscaled everywhere.

The role of Alfvénic decorrelation in anisotropic turbulence has been quantified by measuring the long-wavelength turbulent response in simulations of the drift-Alfvén model.[7] The model is based on reduced MHD,[9] which can be derived as a expansion about the local magnetic field in some region of interest. At the point of expansion the local field is equal to the mean field defined by an average over that region. In the simulations, Alfvénic propagation along the local magnetic field is wholly removed by setting the parallel wave number to zero, yet the decorrelation rate is measured to be Alfvénic. In contrast, the turbulent response of the density decays on the slower eddy straining time scale. The combination of Alfvénic decorrelation in the magnetic and flow evolution and fluid straining decorrelation in the density evolution leads to a spectral index for the electron density of $-7/4$, in close agreement with the observed spectrum for interstellar turbulence. These effects are evident in a statistical closure of the model equations, and are verified in numerical simulation. They contradict the postulate of Goldreich and Sridhar,[10] who posit that for fluctuations whose anisotropy is sufficient to make the Alfvénic decorrelation rate of the mean magnetic field equal to or smaller than the fluid straining rate, the fluid straining rate governs all turbulent decorrelations. They also contradict a broader interpretation of anisotropy that invokes the local field.[11] We emphasize that the reduced description used in reaching our conclusions cannot address the nature of the anisotropy. Rather it shows that for the strongest possible anisotropy, the Alfvén effect is present. We also emphasize that at the heuristic level of Fig. 1 the Alfvén effect is present, independent of the reduced MHD approximation used to construct the simulation model. The numerical measurement of a 5/3 spectral index in recent three-dimensional (3D) MHD simulations[12] (implying the Alfvénic decorrelation is unimportant) may be an artifact of limited resolution. Noting that the turbulence was reported to be ''practically isotropic,''[12] the simplest explanation of the 5/3 exponent is that there were insufficient modes to resolve a scale where Alfvén waves propagating on a larger scale field fluctuation can achieve an Alfvén frequency much bigger than an eddy turnover rate.

In this paper, nonlinear interactions of the drift-Alfvén model are analyzed using strong-turbulence statistical closure theory and computation. It is not possible to consider every aspect of the model in a single work. We focus here on nonlinear properties and leave other effects, including dissipation, diamagnetic drifts, and kinetic effects, to future consideration. These are important in interstellar turbulence and the high frequency magnetic turbulence of MST. Hence this is a study of basic nonlinear properties more than a study of either application. Closure theory provides the characteristic temporal responses of the dominant nonlinear interactions and their associated eigenmodes. This analysis identifies the dominant coupling at long wavelengths as Alfvénic, and indicates that there is an equipartition of kinetic and magnetic energies, with internal energy decoupled from energy exchanges between fields. At short wavelengths, an interaction between the magnetic field and density dominates the Alfvénic coupling. The new interaction resides in the nonlinear coupling of magnetic field and density through the field-aligned pressure force and parallel compression of field-aligned current. The compressional coupling of electron density and magnetic field produces an Alfvén-like oscillation that propagates along the field and is the fluid realization of a fluctuation known as the kinetic Alfvén wave. In turbulence, kinetic Alfvén disturbances propagate as nonlinear wave packets along the perturbed field associated with each scale in the spectrum. In the kinetic Alfvén disturbance electron pressure fluctuations replace the flow in stretching magnetic field lines. The result is an equipartition of magnetic and internal energy, with flow assuming a subdominant role in the dynamics. As apparent from simulations, equipartition of magnetic and internal energy occurs even when the system is driven solely through the magnetic field and the density is subjected to no external forcing. In such cases the kinetic energy is small compared to the magnetic and internal energies. When all three fields are externally forced, Kolmogorov-type analysis of the closure equations reveals a similarity range at short wavelength with equipartitioned magnetic and internal energies.

In the short wavelength regime, the spectral index recovered for the power spectrum of electron density is -2. For the kinetic energy spectrum, the fluid straining of vorticity by the flow dominates the Alfvénic interaction, leading to a spectral index of $-5/3$. The spectral index for the power spectrum of density is obviously very close to the value of

-1.9 extracted from the measurements near the inner scale of angular broadening of extragalactic radio sources.[3] The agreement may be fortuitous and should be viewed with caution. The present theory does not consider dissipation, which is also a potential explanation for the steepening of the observed spectrum. Furthermore, the k^{-2} density fluctuation power spectrum is an asymptotic short wavelength spectrum valid when wavelengths become smaller than the gyroradius. Before entering this regime there is complicated transition subrange whose physics is not yet clear.

This paper is organized as follows. In Sec. II the model is introduced, and its properties are briefly discussed. From its basic properties, a heuristic derivation of the spectra is presented. Section III details the closure calculation necessary to justify the assumptions of the heuristic spectrum derivation. Section IV is devoted to similarity range spectra. The closure equations are analyzed to establish dominant balances through consistency checks, and power law spectral indices are obtained for the kinetic, magnetic, and internal energies in both the long and short wavelength limits. This paper is primarily analytical. However, Sec. V briefly presents results of numerical solutions that illustrate the basic physics described in the earlier sections. Section VI contains the conclusions.

II. MODEL EQUATIONS

The drift-Alfvén model links MHD with electron density evolution specified through the continuity equation. The electron pressure in Ohm's law couples the density to the magnetic field evolution and the compression of electron current along the mean magnetic field couples the magnetic field to the density evolution. Formal derivations have been given by Hazeltine[13] and Rahman and Weiland.[14] It is possible to approximate the system under reduced MHD ordering, appropriate to low beta and a spectrum in which variations along the local magnetic field are weaker than those across the field. While beta, the ratio of plasma pressure to magnetic field pressure, is order unity in the warm diffuse ISM, the fluctuations are generally represented as shear-Alfvén disturbances, which do not involve finite beta. Reduced MHD ordering is based on a coordinate system aligned with the local field. The mean field and local fields are equal at the origin, which is an arbitrary point on the field line in some region of interest. The flow and the magnetic field can be expressed in terms of scalar stream functions corresponding to the electrostatic potential and the component of the magnetic vector potential along the mean magnetic field. The nonlinearities describe the interaction of fluctuations whose wave numbers lie in the plane perpendicular to the mean magnetic field. The model equations are

$$\frac{\partial \hat{\psi}}{\partial t}+\nabla_{\|}\hat{\phi}=\eta\hat{J}+\nabla_{\|}\hat{n}+\frac{C_s}{V_A}n_0^{-1}\nabla\hat{\psi}\times z\cdot\nabla n_0, \qquad (1)$$

$$\frac{\partial}{\partial t}\nabla_{\perp}^2\hat{\phi}-\nabla\hat{\phi}\times z\cdot\nabla\nabla_{\perp}^2\hat{\phi}=-\nabla_{\|}\hat{J}, \qquad (2)$$

$$\frac{\partial \hat{n}}{\partial t}-\nabla\hat{\phi}\times z\cdot\nabla\hat{n}+\nabla_{\|}\hat{J}-\frac{C_s}{V_A}n_0^{-1}\nabla\hat{\phi}\times z\cdot\nabla n_0=0, \qquad (3)$$

where

$$\nabla_{\|}=\frac{\partial}{\partial z}+\nabla\hat{\psi}\times z\cdot\nabla, \qquad (4)$$

$$\hat{J}=\nabla_{\perp}^2\hat{\psi}, \qquad (5)$$

and $\hat{\psi}=(C_s/c)eA_z/T_e$ is the normalized component of the electromagnetic vector potential parallel to the mean magnetic field, $\hat{\phi}=(C_s/V_A)e\phi/T_e$ is the normalized electrostatic potential, $\hat{n}=(C_s/V_A)\tilde{n}/n_0$ is the normalized electron density fluctuation, $\eta=(c^2/4\pi V_A\rho_s)\eta_{sp}$ is the normalized resistivity with η_{sp} the Spitzer resistivity, n_0 is an inhomogeneous mean density, and z is the direction of the mean magnetic field. In the above equations, spatial scales are normalized to $\rho_s=C_s/\Omega_i$ (the ion gyroradius evaluated at the electron temperature), time is normalized to the Alfvén time $\tau_A=\rho_s/V_A$, $C_s=(T_e/m_i)^{1/2}$ is the ion acoustic velocity, $V_A=B/(4\pi m_i n_0)^{1/2}$ is the Alfvén velocity, and $\Omega_i=eB/m_i c$ is the ion gyrofrequency. We have selected a normalization for which all nonlinearities have equal strength when the normalized field amplitudes are equal. The ordering scheme leading to Eqs. (1)–(5) makes no assumption about fluctuation scale, i.e., the model applies to scales both larger and smaller than the gyroradius.

Equations (1)–(3) readily reduce to other familiar systems. If the electron density is eliminated, Eqs. (1) and (2) become reduced MHD (RMHD).[9] If the magnetic field fluctuations are negligible, and the density satisfies the Boltzmann relation ($\hat{n}=\hat{\phi}$), Eqs. (2) and (3) can be combined to yield the Hasegawa–Mima equation for drift wave turbulence.[15] If the electrostatic potential becomes negligible, Eqs. (1) and (3) become the compressible electron magnetoturbulence (CEMT) model.[16] The dynamics of the full system is naturally attracted to the dynamics of these reduced systems under certain circumstances detailed below. In particular, we consider the limit in which the diamagnetic frequency is smaller than the Alfvén frequency. Then, the reductions to RMHD (with density evolving as a passively advected scalar) and CEMT (with flow as a decoupled self-advected field) emerge in long and short wavelength limits under forcing that preferentially excites the magnetic field fluctuations.

The ordering scheme used to derive Eqs. (1)–(5) assumes that variations along the local field are weaker than those across it. This restricts nonlinear coupling to directions across the field, and hence the spectrum is intrinsically anisotropic. We impose a further anisotropy constraint by setting to zero the derivative with respect to z. This is tantamount to stipulating, as did Goldreich and Sridhar,[10] that in a three-dimensional (3D) spectrum the opposing propagation of Alfvénic disturbances along the mean magnetic field play no role in the turbulent decorrelation. Consequently, our assumptions approximate 3D turbulence lying within the critical balance envelope of Ref. 10, i.e., $k_z \lesssim k_{\perp}^{2/3}L^{-1/3}$, where L is the outer scale. Note however, that the Alfvén effect can in no way be considered as removed from Eqs. (1)–(5), because there remains the fluctuating part of the magnetic field, which is perpendicular to the mean magnetic field. Alfvénic disturbances propagate on these field components as wiggles on wiggles, competing with the fluid straining interaction in setting the turbulent decorrelation rate. Although the anisotropic model used herein is consistent with the postulates of Ref. 10, the model, and the conclusions that follow therefrom, may not apply uniformly to 3D interstellar turbulence. For example, interstellar turbulence may be intermittent, having regions where the magnetic field is coherent and characterized by a single scale or narrow range of scales. For the local field in such regions, the nonlinear Alfvén effect of a multiscaled field discussed above is eliminated. However, such regions alone cannot reproduce the broad spectrum, and there must be other regions where the field is multiscaled and the model of Eqs. (1)–(5) is applicable.

In the competition between Alfvénic and eddy straining effects, the Alfvénic disturbances have the advantage of nonlocality. At any given inertial range wave number k, the most rapid Alfvénic decorrelation is produced by fluctuations propagating in opposite directions along a large scale field fluctuation. In contrast, the most rapid fluid straining decorrelation involves the interaction of eddies with scales comparable to k. If the flow and magnetic fields are equipartitioned at k, and the spectrum peaks at small k, then the fluid straining decorrelation is slower than the Alfvénic decorrelation because the large scale field on which the fluctuations propagate is larger than the local flow. To examine the character of the fastest nonlocal Alfvénic interaction in expressions for the turbulent inertial range fields, it is advantageous to separate it from weaker, more local interactions. Fourier transforming Eqs. (1)–(3) and introducing this separation yields

$$\frac{\partial\hat{\psi}_k}{\partial t}-\mathbf{k}_0\times\mathbf{z}\cdot\mathbf{k}\hat{\psi}_{k_0}\hat{\phi}_{k-k_0}+\mathbf{k}_0\times\mathbf{z}\cdot\mathbf{k}\hat{\psi}_{k_0}\hat{n}_{k-k_0}-ik_y\hat{\psi}_k\frac{C_s}{V_AL_n}$$
$$=\sum_{k'\neq k_0}(\mathbf{k}'\times\mathbf{z}\cdot\mathbf{k})[\hat{\psi}_{k'}\hat{\phi}_{k-k'}-\hat{\psi}_k\hat{n}_{k-k'}]=N_\psi, \tag{6}$$

$$(k-k_0)^2\frac{\partial\hat{\phi}_{k-k_0}}{\partial t}+\mathbf{k}_0\times\mathbf{z}\cdot\mathbf{k}\hat{\psi}_{-k_0}\hat{\psi}_k k^2$$
$$=-\frac{1}{2}\sum_{k'}(\mathbf{k}'\times\mathbf{z}\cdot\mathbf{k})[(k-k')^2-k'^2]\hat{\phi}_k\cdot\hat{\phi}_{k-k'}$$
$$+\frac{1}{2}\sum_{k'\neq k_0}(\mathbf{k}'\times\mathbf{z}\cdot\mathbf{k})[(k-k')^2-k'^2]\hat{\psi}_{k'}\hat{\psi}_{k-k'}\equiv N_\phi, \tag{7}$$

$$\frac{\partial\hat{n}_{k-k_0}}{\partial t}-\mathbf{k}_0\times\mathbf{z}\cdot\mathbf{k}\hat{\psi}_{-k_0}\hat{\psi}_k k^2-i(k_y-k_{0y})\hat{\phi}_{k-k_0}\frac{C_s}{V_AL_n}$$
$$=\sum_{k'}(\mathbf{k}'\times\mathbf{z}\cdot\mathbf{k})\hat{\phi}_{k'}\hat{n}_{k-k'}+\frac{1}{2}\sum_{k'\neq k_0}(\mathbf{k}'\times\mathbf{z}\cdot\mathbf{k})$$
$$\times[(k-k')^2-k'^2]\hat{\psi}_{k'}\hat{\psi}_{k-k'}\equiv N_n. \tag{8}$$

In writing Eqs. (6)–(8), $\mathbf{k}_0$ is understood to represent not a single wave number but the wave numbers of a restricted band corresponding to the most energetic modes near the outer scale of the magnetic fluctuation spectrum. The equations are written to isolate the interaction of nonlocal triplets involving modes at $\mathbf{k}$, $\mathbf{k}_0$, and $\mathbf{k}-\mathbf{k}_0$ with $k_0\ll k$. To better see the wave number scalings of these nonlocal Alfvénic couplings, we express Eqs. (6)–(8) terms of the fields $|\mathbf{B}_k|=B_k=k|\psi_k|$ and $|\mathbf{V}_k|=V_k=k\phi_k$. The equations then take the heuristic form

$$\frac{\partial B_k}{\partial t}-\gamma_AV_k+\gamma_Akn_k=\omega_*B_k+kN_\psi, \tag{9}$$

$$\frac{\partial V_k}{\partial t}+\gamma_AB_k=\frac{N_\phi}{k}, \tag{10}$$

$$\frac{\partial n_k}{\partial t}-\gamma_AkB_k=\omega_*\frac{B_k}{k}+N_n, \tag{11}$$

where $\gamma_A=|\mathbf{k}_0\times\mathbf{z}\cdot\mathbf{k}\hat{\psi}_{k_0}|$ and $\omega_*=k_yC_s/V_AL_n$. The nonlocal Alfvénic interactions have been grouped on the left hand side; all other interactions are on the right hand side.

From these equations we readily deduce the following general behavior:

(1) At long wavelengths ($k_i<k\ll 1$, where k_i is the driving scale), the off-diagonal coupling of V_k and B_k (with coupling strength γ_A) is the dominant nonlocal interaction. This is the standard Alfvénic interaction of MHD. The off-diagonal character yields counter-propagating Alfvénic disturbances with equipartitioned magnetic and flow fields. The frequency of propagation is $\gamma_A\approx kV_A\approx kk_0\psi_{k_0}$.

(2) At short wavelengths ($k>k_i\gg 1$), the off-diagonal coupling of B_k and n_k (with coupling strength $|\gamma_A|k$) is the dominant nonlocal interaction. This is the kinetic Alfvén disturbance. It is similar to an Alfvén wave because the field coupling is off-diagonal, leading to counter propagating disturbances along the large scale field $k_0\psi_{k_0}$, with equipartitioned density and magnetic field. The frequency of propagation is $\gamma_Ak\approx k^2V_A$.

(3) In both limits, both types of the interactions occur, but the additional factor of k in the kinetic Alfvén coupling makes it dominant in the short wavelength regime and subdominant in the long wavelength regime.

(4) Provided the Alfvénic disturbances govern the turbulent decorrelation, the spectrum in the long wavelength regime is the Iroshnikov–Kraichnan spectrum.[8,17] A heuristic derivation is easily obtained from the long wavelength spectrum balances that are formed by casting Eqs. (9) and (10) as energy evolution equations and invoking a Kolmogorov balance between a constant energy dissipation rate ϵ and the dominant nonlinearity. For magnetic energy B^2, Eq. (9) yields

$$\epsilon=B\frac{dB}{dt}\cong BkN_\psi\cong kB^2V, \tag{12}$$

while for kinetic energy V^2, Eq. (10) yields

$$\epsilon=V\frac{dV}{dt}\cong\frac{VN_\phi}{k}\cong kB^2V, \tag{13}$$

where all fields are taken to lie in the vicinity of the inertial range wave number k. We substitute singly for any one member of the triplet B^2V from an inversion of the appropriate amplitude equation and assume equipartition of V and B. For example, the steady state inversion of B is specified from Eq. (10) as

$$B \approx \frac{N_\phi}{k\gamma_A} \approx \frac{kB^2}{\gamma_A}. \tag{14}$$

Note that, provided γ_A evolves slowly relative to N_ϕ, B is driven by the turbulent interaction of the flow equation. The energy balances then become $\epsilon = k^2B^4/\gamma_A = kB^4/V_A$. The spectra of the equipartitioned magnetic and flow fields are thus

$$E_M(k) = E_K(k) = \epsilon^{1/2}V_A^{1/2}\frac{1}{k^{3/2}}, \tag{15}$$

where $E_M(k) \equiv B_k^2/k$ and $E_K(k) \equiv V_k^2/k$.

(5) A similar derivation produces the short wavelength spectra. The spectrum balances associated with the dominant kinetic-Alfvén nonlinearities are

$$\epsilon = B\frac{dB}{dt} \cong BkN_\psi \cong k^2B^2n, \tag{16}$$

$$\epsilon = n\frac{dn}{dt} \cong nN_n \cong k^2B^2n. \tag{17}$$

We substitute singly for any one member of the triplet B^2n using the appropriate amplitude equation and assume equipartition of B and n. For the short wavelength limit, the steady state inversion of B is specified from Eq. (11) as

$$B \approx \frac{N_n}{\gamma_A k} \approx \frac{k^2B^2}{\gamma_A k}. \tag{18}$$

The energy balances then become $\epsilon = k^4B^4/\gamma_A k = k^2B^4/V_A$. The spectra of the equipartitioned magnetic and density fields thus satisfy

$$E_M(k) = E_I(k) = \epsilon^{1/2}V_A^{1/2}k^{-2}, \tag{19}$$

where $E_I \equiv n_k^2/k$. The slightly steeper decay law is the consequence of the additional factor of k in the kinetic-Alfvén coupling.

The heuristic derivations in points (4) and (5) above rely on an inversion of the amplitude equations, given in Eqs. (14) and (18), to determine the correlation time of the nonlinear interactions. A similar step also occurs in more formal closure calculations, like that of the next section. In such inversions it is essential to account for the fact that disturbances at wave number k decorrelate on a time scale kV_A that is short relative to the time scale of the long wavelength field $k_0\hat{\psi}_{k_0}$ on which the disturbances propagate. Because the long wavelength field forms a quasistationary background on the time scale of inertial range correlations, the off-diagonal character of the nonlocal Alfvénic coupling effectively mixes the turbulent sources [so that B is driven by N_ϕ or N_n, as in Eq. (14) or (18)]. It is difficult to account for this mixing unless the time scale separation is explicitly treated by separating the local and nonlocal interactions as done in Eqs. (6)–(8). The time scale separation holds provided the spectrum peaks at long wavelength.

The heuristic derivations above rely on the assumption that the nonlocal Alfvénic interactions propagating along the long wavelength magnetic field fluctuations dominate the decorrelation of the triplet correlations governing spectral transfer. This assumption is tested and validated by systematically performing a statistical closure calculation for strong turbulence, and assessing the relative role of the Alfvénic and fluid straining interactions.

III. CLOSURE CALCULATION

We apply the eddy damped quasinormal Markovian (EDQNM) closure procedure[18] to the basic evolution equations in wave number space. The closure allows identification of the correlations and couplings that drive spectral transfer in a system complicated by an active density response that becomes important at small wavelength. It also sorts out the interplay between the rapid Alfvénic and the slow fluid straining time scales. Which time scale governs turbulent decorrelation has been a source of confusion even in the simpler situation where density is passive. The basic equations are

$$\frac{\partial\hat{\psi}_k}{\partial t} - \omega_A\hat{\phi}_k + \omega_A\hat{n}_k - i\omega_*\hat{\psi}_k = S_\psi, \tag{20}$$

$$\frac{\partial k^2\hat{\phi}_k}{\partial t} + \gamma_\phi k^2\hat{\phi}_k + \hat{\omega}_A k^2\hat{\psi}_k = S_\phi, \tag{21}$$

$$\frac{\partial\hat{n}_k}{\partial t} + \gamma_n\hat{n}_k - \hat{\omega}_A k^2\hat{\psi}_k - i\omega_*\hat{\phi}_k = S_n, \tag{22}$$

where

$$S_\psi = \sum_{k'} A_{k',k}[\hat{\psi}_{k'}(\hat{\phi}_{k-k'} - \hat{n}_{k-k'}) - \hat{\psi}_{k-k'}(\hat{\phi}_{k'} - \hat{n}_{k'})], \tag{23}$$

$$S_\phi = \sum_{k'} A_{k',k}[(k-k')^2 - k'^2](\hat{\psi}_{k'}\hat{\psi}_{k-k'} - \hat{\phi}_{k'}\hat{\phi}_{k-k'}), \tag{24}$$

$$S_n = -\sum_{k'} A_{k',k}\{\hat{\phi}_{k'}\hat{n}_{k-k'} - \hat{n}_{k'}\hat{\phi}_{k-k'} + [(k-k')^2 - k'^2]\hat{\psi}_{k'}\hat{\psi}_{k-k'}\}, \tag{25}$$

and $A_{k',k} = (1/2)(\mathbf{k}' \times \mathbf{z} \cdot \mathbf{k})$. The quantities ω_A, $\hat{\omega}_A$, γ_n, and γ_ϕ are complex-valued, amplitude dependent turbulent decorrelation rates whose form is to be derived self-consistently from the closure calculation. The basic structure, with ω_A and $\hat{\omega}_A$ as off-diagonal coupling coefficients, is motivated by the nonlocal Alfvénic decorrelation discussed in the preceding section. Those considerations lead to the prediction that $\omega_A \approx \hat{\omega}_A \approx \gamma_A \approx kB_{k_0}$. The two Alfvénic decorrelation rates ω_A and $\hat{\omega}_A$ allow for possible differences in the responses of the vorticity and density relative to that of the magnetic potential. The rates γ_ϕ and γ_n are fluid straining decorrelations associated with the advection of vorticity and density by the flow. These are treated in the usual manner as diagonal diffusivities. Dimensional considerations and the locality of the fluid straining decorrelation[8] lead to a prediction that $\gamma_\phi \approx \gamma_n \approx k V_k$.

To identify and extract the time scales and couplings of the fundamental interactions in Eqs. (20)–(22) we consider the eigenmodes of the left hand side of these equations. We anticipate that the decorrelation rates ω_A, $\hat{\omega}_A$, γ_n, and γ_ϕ are spectrum averaged quantities that evolve in a stationary state on a slower time scale than the mode amplitudes $\hat{\psi}_k$, $\hat{\phi}_k$, and $\hat{n}_k$. Hence the rates can be treated as constants on the faster time scale, even though they are functions of amplitude. For a normal mode response $\hat{\chi}_k = \chi_0 \exp(-i\omega t)$, where $\hat{\chi}_k$ is any of the fluctuation amplitudes, the characteristic equation of the left hand side of Eqs. (20)–(22) is

$$\omega^3 - i\omega^2(\gamma_n + \gamma_\phi - i\omega_*) - \omega[\omega_A\hat{\omega}_A(1+k^2) + \gamma_n\gamma_\phi - i\omega_*(\gamma_n+\gamma_\phi)] - i\omega_A\hat{\omega}_A(\gamma_n + \gamma_\phi k^2) - \omega_*(\omega_A\hat{\omega}_A + \gamma_n\gamma_\phi) = 0. \tag{26}$$

Assuming an ordering $\omega_A \sim \hat{\omega}_A \gg \gamma_n \sim \gamma_\phi \sim \omega_*$, the three roots are given by

$$\omega_{1,2} = \pm[\omega_A\hat{\omega}_A(1+k^2)]^{1/2} + O(\gamma_n), \tag{27}$$

$$\omega_3 = -i[\gamma_n + \gamma_\phi k^2] - \omega_*/(1+k^2) + O(\gamma_n^2/\omega_A). \tag{28}$$

The paired roots $\omega_{1,2}$ represent the Alfvénic interaction, corresponding to disturbances propagating in opposite directions along the large scale magnetic field fluctuation. For short wavelengths this branch clearly remains Alfvénic, but the eigenfrequency changes from ω_A to $k\omega_A$. The third root ω_3 represents a decorrelation process tied to the fluid straining or eddy turnover time scale. This branch is designated as the arbiter of turbulent decorrelation rate in theories whose spectral index is given by 5/3.[10] In actuality, the spectral transfer rates are a mixture of the three eigenmodes associated with the eigenfrequencies ω_1, ω_2, and ω_3. Only if the mixture has a negligible contribution from the Alfvénic branches will the fluid straining rate set the decorrelation rate. Measurement of this mixture in long wavelength simulations has established that the nonlinearities of the magnetic and electrostatic potential equations are dominated by the Alfvénic eigenmodes.[7] Consequently the Alfvénic decorrelation dominates the spectral transfer.[1]

The eigenvectors corresponding to the Alfvénic roots $\omega_{1,2}$ of Eq. (27) are

$$\hat{\psi}_k = \hat{\psi}_0, \tag{29}$$

$$\hat{\phi}_k = -i\frac{\omega_A}{\omega_{1,2}}\hat{\psi}_O = \mp i\left[\frac{\hat{\omega}_A}{\omega_A(1+k^2)}\right]^{1/2}\hat{\psi}_O, \tag{30}$$

$$\hat{n}_k = i\frac{\hat{\omega}_A k^2}{\omega_{1,2}}\hat{\psi}_O = \pm i\left[\frac{\hat{\omega}_A}{\omega_A(1+k^2)}\right]^{1/2} k^2\hat{\psi}_O. \tag{31}$$

For $k \ll 1$ these eigenvectors have comparable amplitudes in $\hat{\psi}_k$ and $\hat{\phi}_k$ (and hence B and V), and reflect the equipartition of Alfvénic turbulence. The density $\hat{n}_k$ is smaller than $\hat{\psi}_k$ and $\hat{\phi}_k$ by a factor k^2 (smaller than B and V by a factor k). For $k \gg 1$, the eigenmode partition changes noticeably. The density is proportional to $k\hat{\psi}_O$ and is therefore in equipartition with the magnetic field. The potential $\hat{\phi}_k$ is smaller than $\hat{\psi}_k$ by the factor k^{-1} (V is smaller than B by the same factor). It is clear that when the frequency of Alfvénic interactions changes from ω_A to $\omega_A k$, the dominant coupling changes from one in which the energy exchange drives equipartition of B and V to one in which the B and n become equipartitioned. This reflects the change from the shear Alfvén interaction at long wavelength to the kinetic Alfvén interaction at short wavelength. The eigenvector of the fluid straining branch is most simply illustrated for ω_* negligible:

$$\hat{\psi}_k = \hat{\psi}_O, \tag{32}$$

$$\hat{\phi}_k = \frac{\hat{\omega}_A}{i\omega_3 - \gamma_\phi}\hat{\psi}_O = \frac{\hat{\omega}_A(1+k^2)}{\gamma_n - \gamma_\phi}\hat{\psi}_O, \tag{33}$$

$$\hat{n}_k = \frac{-\hat{\omega}_A k^2}{i\omega_3 - \gamma_n}\hat{\psi}_O = \frac{\hat{\omega}_A(1+k^2)}{\gamma_n - \gamma_\phi}\hat{\psi}_O. \tag{34}$$

For this branch $\hat{n}_k$ and $\hat{\phi}_k$ are equal and larger than $\hat{\psi}_k$. If fluid straining were the dominant nonlinear decorrelation, the amplitudes could be expected to partition according to Eqs. (32)–(34). Indeed this is the characteristic partition of electrostatic drift wave turbulence.

We proceed with the closure, forming spectrum energy equations from Eqs. (20)–(22) by multiplying each equation, respectively, by $k^2\hat{\psi}_{-k}$, $\hat{\phi}_{-k}$, and $\hat{n}_{-k}$, and taking the complex conjugate,

$$\frac{\partial k^2|\hat{\psi}_k|^2}{\partial t} - 2k^2\,\mathrm{Re}[\omega_A\langle\hat{\phi}_k\hat{\psi}_{-k}\rangle - \omega_A\langle\hat{n}_k\hat{\psi}_{-k}\rangle] = 2k^2\,\mathrm{Re}[\langle S_\psi\hat{\psi}_{-k}\rangle], \tag{35}$$

$$\frac{\partial k^2|\hat{\phi}_k|^2}{\partial t} - 2\,\mathrm{Re}[-\gamma_\phi k^2|\hat{\phi}_k|^2 - \hat{\omega}_A k^2\langle\hat{\psi}_k\hat{\phi}_{-k}\rangle] = 2\,\mathrm{Re}[\langle S_\phi\hat{\phi}_{-k}\rangle], \tag{36}$$

$$\frac{\partial|\hat{n}_k|^2}{\partial t} - 2\,\mathrm{Re}[-\gamma_n|\hat{n}_k|^2 + \hat{\omega}_A k^2\langle\hat{\psi}_k\hat{n}_{-k}\rangle] = 2\,\mathrm{Re}[\langle S_n\hat{n}_{-k}\rangle]. \tag{37}$$

To determine the lifetime of the source correlations making up the last terms of Eqs. (35)–(37), the turbulent evolution equation for each fluctuation appearing in those correlations must be iteratively solved. Anticipating the iteration, the source correlations take the form

$$\langle S_\psi\hat{\psi}_{-k}\rangle = \sum_{k'} A_{k',k}\{2\langle\hat{\psi}_{k'}(\hat{\phi}^{(2)}_{k-k'} - \hat{n}^{(2)}_{k-k'})\hat{\psi}_{-k} - 2\hat{\psi}^{(2)}_{k-k'} \times(\hat{\phi}_{k'} - \hat{n}_{k'})\hat{\psi}_{-k}\rangle + [\langle\hat{\psi}_{k'}(\hat{\phi}_{k-k'} - \hat{n}_{k-k'}) - \hat{\psi}_{k-k'}(\hat{\phi}_{k'} - \hat{n}_{k'})]\hat{\psi}^{(2)}_{-k}\rangle\}, \tag{38}$$

$$\langle S_\phi \hat{\phi}_{-k}\rangle = \sum_{k'} A_{k',k}[(k-k')^2 - k'^2]\{2\langle \hat{\psi}_{k'} \hat{\psi}^{(2)}_{k-k'} \hat{\phi}_{-k}\rangle - 2\langle \hat{\phi}_{k'} \hat{\phi}^{(2)}_{k-k'} \hat{\phi}_{-k}\rangle + [\langle \hat{\psi}_{k'} \hat{\psi}_{k-k'} - \hat{\phi}_{k-k'} \hat{\phi}_{k'}]\hat{\phi}^{(2)}_{-k}\rangle\}, \quad (39)$$

$$\langle S_n \hat{n}_{-k}\rangle = -\sum_{k'} A_{k',k}\{2\langle[\hat{\phi}_{k'} \hat{n}^{(2)}_{k-k'} - \hat{n}_{k'} \hat{\phi}^{(2)}_{k-k'} + [(k-k')^2 - k'^2]\hat{\psi}_{k'} \hat{\psi}^{(2)}_{k-k'}]\hat{n}_{-k}\rangle + \langle[\hat{\phi}_{k'} \hat{n}_{k-k'} - \hat{\phi}_{k-k'} \hat{n}_{k'}]\hat{n}^{(2)}_{-k}\rangle + [(k-k')^2 - k'^2]\langle \hat{\psi}_{k'} \hat{\psi}_{k-k'} \hat{n}^{(2)}_{-k}\rangle\}, \quad (40)$$

where the fluctuations with the superscript (2) are those for which inversions of the amplitude equations, Eqs. (20)–(22), are substituted. The inversions are performed formally, with the right hand sides treated as sources. Moreover, the Fourier modes contributing to each source in Eqs. (23)–(25) are restricted to the directly interacting triplets which close fourth order correlations as products of second order correlations. Thus the fields $\hat{\psi}^{(2)}_{k-k'}$, $\hat{\phi}^{(2)}_{k-k'}$, and $\hat{n}^{(2)}_{k-k'}$ satisfy

$$\frac{\partial \hat{\psi}^{(2)}_{k-k'}}{\partial t} - \omega''_A \hat{\phi}^{(2)}_{k-k'} + \omega''_A \hat{n}^{(2)}_{k-k'} - i\omega''_* \hat{\psi}^{(2)}_{k-k'} = S^{(2)}_\psi(k-k'), \quad (41)$$

$$(k-k')^2 \frac{\partial \hat{\phi}^{(2)}_{k-k'}}{\partial t} + \gamma''_\phi (k-k')^2 \hat{\phi}_{k-k'} + \hat{\omega}''_A (k-k')^2 \hat{\psi}_{k-k'} = S^{(2)}_\phi(k-k'), \quad (42)$$

$$\frac{\partial \hat{n}^{(2)}_{k-k'}}{\partial t} + \gamma''_n \hat{n}^{(2)}_{k-k'} - \hat{\omega}''_A (k-k')^2 \hat{\psi}^{(2)}_{k-k'} - i\omega''_* \hat{\phi}^{(2)}_{k-k'} = S^{(2)}_n(k-k'), \quad (43)$$

where the superscript $''$ denotes the evaluation of functions of wave number at $k-k'$, and

$$S^{(2)}_\psi(k-k') = 2A_{k',k}[\hat{\psi}_k(\hat{\phi}_{-k'} - \hat{n}_{-k'}) - \hat{\psi}_{-k'}(\hat{\phi}_k - \hat{n}_k)], \quad (44)$$

$$S^{(2)}_\phi(k-k') = 2A_{k',k}(k'^2 - k^2)[\hat{\psi}_k \hat{\psi}_{-k'} - \hat{\phi}_k \hat{\phi}_{-k'}], \quad (45)$$

$$S^{(2)}_n(k-k') = -2A_{k',k}[\hat{\phi}_k \hat{n}_{-k'} - \hat{\phi}_{-k'} \hat{n}_k + (k'^2 - k^2)\hat{\psi}_{-k'} \hat{\psi}_k]. \quad (46)$$

Consistent with the Markovian assumption of EDQNM we seek a steady state inversion of Eqs. (41)–(43). This is valid provided the spectrum evolves on a time scale that is long compared to the correlation times of the turbulent fluctuations. Dropping the time derivatives, Eqs. (41)–(43) yield

$$\hat{\psi}^{(2)}_{k-k'} = \frac{\gamma''_n S^{(2)}_\phi(k-k') - [k-k']^2 \gamma''_\phi S^{(2)}_n(k-k')}{(k-k')^2 \hat{\omega}''_A [\gamma''_n + (k-k')^2 \gamma''_\phi]} + O(\gamma_n/\omega_A^2), \quad (47)$$

$$\hat{\phi}^{(2)}_{k-k'} = \frac{S^{(2)}_\phi(k-k') + S^{(2)}_n(k-k') - (\gamma''_n/\omega''_A) S^{(2)}_\psi(k-k')}{[\gamma''_n + (k-k')^2 \gamma''_\phi]} + O(\gamma_n/\omega_A^2), \quad (48)$$

$$\hat{n}^{(2)}_{k-k'} = \frac{S^{(2)}_\phi(k-k') + S^{(2)}_n(k-k') + (\gamma''_n/\omega''_A)[k-k']^2 S^{(2)}_\psi(k-k')}{[\gamma''_n + (k-k')^2 \gamma''_\phi]} + O(\gamma_n/\omega_A^2), \quad (49)$$

where an expansion based on $\omega_A \gg \gamma_n \sim \gamma_\phi \gg \omega_*$ has been carried out and only the lowest two orders $[O(\gamma_n^{-1})$ and $O(\omega_A^{-1})]$ have been retained. These solutions reflect the basic structure of drift-Alfvén turbulence. The Alfvénic interaction is off-diagonal, whereas the fluid straining interactions are diagonal. Consequently the sources S_ϕ and S_n are grouped in Eqs. (47)–(49). The driving of magnetic fluctuations $\psi^{(2)}$ by S_ϕ and S_n is off-diagonal; hence it decorrelates on the Alfvénic time scale. The fluctuations $n^{(2)}$ and $\phi^{(2)}$ are driven by S_ϕ and S_n through a diagonal coupling; hence their decorrelation is on the fluid straining time scale. Because S_ϕ and S_n have magnetic components [see Eqs. (44)–(46)], these must cancel out of the diagonal fluid straining term, indicating that S_ϕ and S_n appear in the density and potential equations as a sum. Additionally, potential and density are coupled off-diagonally by the magnetic interaction, so these fluctuations are driven by S_ψ with an Alfvénic decorrelation. The source S_ψ, which is diagonal in the magnetic equation, can only enter through a coupling on the fluid straining time scale. Therefore, there is no S_ψ contribution in the magnetic equation. Finally, $n^{(2)}$ and $\phi^{(2)}$ are equal to lowest order. This means that $n^{(2)}$ and $\phi^{(2)}$ cancel to lowest order in Eq. (38) for $\langle S_\psi \psi_{-k}\rangle$. Since the decorrelation of $\psi^{(2)}$ takes place on the fast time scale only, the magnetic equation, of which $\langle S_\psi \psi_{-k}\rangle$ is the source, has no decay on the fluid straining time scale. This feature is readily apparent in Eq. (50) below.

The final form of the energy evolution equations are obtained by substituting Eqs. (47)–(49) into Eqs. (38)–(40), along with the corresponding inversions of $\hat{\psi}^{(2)}_{-k}$, $\hat{\phi}^{(2)}_{-k}$, and $\hat{n}^{(2)}_{-k}$, which have the same form as Eqs. (47)–(49) with $k-k' \to -k$. The sources $S^{(2)}_\psi(-k)$, $S^{(2)}_\phi(-k)$, and $S^{(2)}_n(-k)$ are given by Eqs. (44)–(46), with $k' \to k-k'$ and

$k\rightarrow -k'$. Under the above substitutions, the source correlations given by Eqs. (38)–(40) now contain the Alfvénic and fluid straining decorrelation terms explicitly displayed in Eqs. (35)–(37). Consequently the energy evolution equations are simply

$$\frac{\partial k^2|\hat{\psi}_k|^2}{\partial t}=2k^2\,\mathrm{Re}\langle S_\psi\hat{\psi}_{-k}\rangle,$$

$$\frac{\partial k^2|\hat{\phi}_k|^2}{\partial t}=2\,\mathrm{Re}\langle S_\phi\hat{\phi}_{-k}\rangle,$$

$$\frac{\partial|\hat{n}_k|^2}{\partial t}=2\,\mathrm{Re}\langle S_n\hat{n}_{-k}\rangle.$$

The right hand side of these equations will be examined to extract those terms having the structure of the Alfvénic and fluid straining decorrelation terms in Eqs. (35)–(37). This procedure then yields the expressions for ω_A, $\hat{\omega}_A$, γ_n, and γ_ϕ. In carrying out the substitutions, we note that $\hat{\phi}^{(2)}$ and $\hat{n}^{(2)}$ [Eqs. (48) and (49)] each contain terms of order γ^{-1}, and a term of order ω_A^{-1} that is proportional to $S_\psi^{(2)}$. Normally the terms of order ω_A^{-1} are higher order and can be neglected. However in Eq. (38), the lowest order components of $\hat{\phi}^{(2)}$ and $\hat{n}^{(2)}$ cancel, and the higher order terms must be retained to recover the form of the Alfvénic decorrelation ω_A. This necessitates the inclusion of the higher order terms in Eqs. (39) and (40) to maintain energy conservation. The energy evolution equations have a rich mathematical structure. They are valid in both the long and short wavelength limits. The exchanges of energy between these equations depends on six separate correlations representing, respectively, the magnetic, kinetic, and internal spectral energy densities $k^2|\hat{\psi}_k|^2$, $k^2|\hat{\phi}_k|^2$, and $|\hat{n}_k|^2$, and cross correlations $\langle\hat{\psi}_k\hat{\phi}_{-k}\rangle$, $\langle\hat{\psi}_k\hat{n}_{-k}\rangle$, and $\langle\hat{n}_k\hat{\phi}_{-k}\rangle$. These exchanges mix the individual energy densities in a process that combines the long time scale and short time scale responses of Eqs. (27) and (28).

The energy evolution equations are given by

$$\frac{\partial k^2|\hat{\psi}_k|^2}{\partial t}=-2\,\mathrm{Re}\sum_{k'}4A_{k',k}^2k^2\{G_A(k,k')-G_{A\phi}(-k',-k)+G_{An}(-k',-k)+G_{A\phi}(-k+k',k')-G_{An}(-k+k',k')\},\tag{50}$$

$$\frac{\partial k^2|\hat{\phi}_k|^2}{\partial t}=2\,\mathrm{Re}\sum_{k'}2A_{k',k}^2[(k-k')^2-k'^2]\times\Big\{2G_{A\phi}(k,k')+2G_{\phi\phi}(k,k')+G_{\phi\phi}(-k',k-k')-G_\psi(-k',k-k')-\frac{\gamma_n(-k)}{[\gamma_n(-k)+k^2\gamma_\phi(-k)]}\times G_A(-k',k-k')\Big\},\tag{51}$$

$$\frac{\partial|\hat{n}_k|^2}{\partial t}=2\,\mathrm{Re}\sum_{k'}2A_{k',k}^2\Big\{2G_\phi(k,k')+2G_{nn}(k,k')-2G_{nn}(-k',k-k')+[(k-k')^2-k'^2]\times\Big[-2G_{An}(k,k')+G_\psi(-k',k-k')-\frac{k^2\gamma_\phi(-k)}{[\gamma_n(-k)+k^2\gamma_\phi(-k)]}G_A(-k',k-k')\Big]\Big\},\tag{52}$$

where

$$G_A(k,k')=\left(\frac{|\hat{\psi}_k|^2[\langle\hat{\phi}_{-k'}\hat{\psi}_{k'}\rangle-\langle\hat{n}_{-k'}\hat{\psi}_{k'}\rangle]-|\hat{\psi}_{k'}|^2[\langle\hat{\phi}_k\hat{\psi}_{-k}\rangle-\langle\hat{n}_k\hat{\psi}_{-k}\rangle]}{\omega_A''}\right),\tag{53}$$

$$G_{A\phi}(k,k')=\frac{\gamma_n''(k'^2-k^2)}{\hat{\omega}_A''(k-k')^2}\left(\frac{|\hat{\psi}_{k'}|^2\langle\hat{\phi}_{-k}\hat{\psi}_k\rangle-\langle\hat{\phi}_{-k'}\hat{\psi}_{k'}\rangle|\hat{\phi}_k|^2}{[\gamma_n''+(k-k')^2\gamma_\phi'']}\right)+\frac{\gamma_\phi''}{\hat{\omega}_A''}\left(\frac{(k'^2-k^2)|\hat{\psi}_{k'}|^2\langle\hat{\psi}_k\hat{\phi}_{-k}\rangle+\langle\hat{n}_{-k'}\hat{\psi}_{k'}\rangle|\hat{\phi}_k|^2-\langle\hat{\phi}_{-k}\hat{n}_k\rangle\langle\hat{\psi}_{k'}\hat{\phi}_{-k'}\rangle}{[\gamma_n''+(k-k')^2\gamma_\phi'']}\right),\tag{54}$$

$$G_{An}(k,k')=\frac{\gamma_n''}{\hat{\omega}_A''}\frac{(k'^2-k^2)}{(k-k')^2}\left(\frac{|\hat{\psi}_{k'}|^2\langle\hat{n}_{-k}\hat{\psi}_k\rangle-\langle\hat{\phi}_{-k'}\hat{\psi}_{k'}\rangle\langle\hat{\phi}_k\hat{n}_{-k}\rangle}{[\gamma_n''+(k-k')^2\gamma_\phi'']}\right)+\frac{\gamma_\phi''}{\hat{\omega}_A''}\left(\frac{(k'^2-k^2)|\hat{\psi}_{k'}|^2\langle\hat{\psi}_k\hat{n}_{-k}\rangle-\langle\hat{\phi}_{-k'},\hat{\psi}_{k'}\rangle|\hat{n}_k|^2+\langle\hat{n}_{-k}\hat{\phi}_k\rangle\langle\hat{\psi}_{k'}\hat{n}_{-k'}\rangle}{[\gamma_n''+(k-k')^2\gamma_\phi'']}\right),\tag{55}$$

$$G_\psi(-k',k-k')=\left(\frac{[(k-k')^2-k'^2][\langle\hat{\phi}_{-k'}\hat{\psi}_{k'}\rangle\langle\hat{\phi}_{-k+k'}\hat{\psi}_{k-k'}\rangle]+2\langle\hat{\psi}_{k-k'}\hat{n}_{-k+k'}\rangle\langle\hat{\psi}_{k'}\hat{\phi}_{-k'}\rangle}{[\gamma_n(-k)+k^2\gamma_\phi(-k)]}\right),\tag{56}$$

$$G_{\phi}(k,k')=\left(\frac{-(k'^2-k^2)\langle\hat{\phi}_k\hat{n}_{-k}\rangle\langle\hat{n}_{k'}\hat{\phi}_{-k'}\rangle+\langle\hat{n}_{k'}\hat{\phi}_{-k'}\rangle|\hat{n}_k|^2-\langle\hat{\phi}_k\hat{n}_{-k}\rangle|\hat{n}_{k'}|^2}{[\gamma_n''+(k-k')^2\gamma_\phi'']}\right), \tag{57}$$

$$G_{\phi\phi}(k,k')=\left(\frac{(k'^2-k^2)|\hat{\phi}_k|^2|\hat{\phi}_{k'}|^2+\langle\hat{n}_{-k'}\hat{\phi}_{k'}\rangle|\hat{\phi}_k|^2-\langle\hat{\phi}_{-k}\hat{n}_k\rangle|\hat{\phi}_{k'}|^2}{[\gamma_n''+(k-k')^2\gamma_\phi'']}\right), \tag{58}$$

$$G_{nn}(k,k')=\left(\frac{(k'^2-k^2)\langle\hat{\phi}_k\hat{n}_{-k}\rangle|\hat{\phi}_{k'}|^2+\langle\hat{n}_{-k'}\hat{\phi}_{k'}\rangle\langle\hat{\phi}_k\hat{n}_{-k}\rangle-|\hat{n}_k|^2|\hat{\phi}_{k'}|^2}{[\gamma_n''+(k-k')^2\gamma_\phi'']}\right). \tag{59}$$

Equations (50)–(52) conserve the total energy $\Sigma(k^2|\hat{\psi}_k|^2+k^2|\hat{\phi}_k|^2+|\hat{n}_k|^2)$. The magnetic, kinetic, and internal energies are not conserved separately. Consequently, the conservative nature of Eqs. (50)–(52) is manifest when the three equations are summed and a summation over k is performed. The terms of the magnetic equation, G_A, $G_{A\phi}$, and G_{An}, sum to zero only with their counterparts in the kinetic and internal energy equations. These terms decorrelate on the Alfvénic time scale. They only conserve energy across equations, reflecting the fact that the Alfvénic interactions convert energy from one form (magnetic, kinetic, internal) to another. The two terms $G_{\phi\phi}$ in the kinetic energy equation add to zero upon summation over k. The same is also true for the terms G_{nn} in the internal energy equation. These terms decorrelate on the fluid straining time scale. They conserve energy within single equations, reflecting the fact that the fluid straining interaction can transfer energy spectrally from one scale to another without converting the form of energy. There is one fluid straining term, G_ψ, that involves conversion between the kinetic and internal energies. The term G_ϕ from the density equation vanishes directly upon summation over k. The terms in which the wave number dependence is indicated by the pair $(-k',k-k')$ do not depend on any correlation or spectral energy density at the wave number k. These terms are often referred to as incoherent, or as turbulence production terms. They are required for conservative spectral transfer of energy. This type of term enters in both Alfvénic and fluid straining terms, indicating that both carry energy from one scale to another. However, the Alfvénic terms do so by converting energy, whereas the fluid straining terms need not produce conversion. From statistical mechanics considerations, the spectral energy transfer is from large scale to small scale.[7]

As indicated earlier, spectral transfer in the magnetic energy equation is governed solely by Alfvénic interactions; no term in this equation decorrelates with fluid straining rates γ_n or γ_ϕ (the fluid straining factors in $G_{A\phi}$ and G_{An} effectively cancel). For strongly nonlocal triads, the terms in G_A proportional to $|\hat{\psi}_k|^2$ cancel with corresponding terms in $G_{A\phi}$ and G_{An}, indicating that these terms favor spectral transfer that is local in wave number space. Indeed local energy transfer is favored by Eqs. (50)–(52). Note that every term in the magnetic equation depends on at least one cross correlation. This again reflects the fact that magnetic energy passes from one scale to another only through a conversion to other energy forms. In contrast, there is a term in the kinetic energy equation that depends solely on the potential autocorrelation. This term is involved in the spectral transfer of kinetic energy without conversion to other energy forms. The kinetic and internal energy evolution equations have terms that decay on both the Alfvénic and fluid straining time scales. In the kinetic energy equation the Alfvénic term dominates except near the inner scale. In the internal energy equation the inverse is true and the fluid straining term dominates except near the inner scale. The details of these balances require the forms of ω_A, $\hat{\omega}_A$, γ_n, and γ_ϕ and the spectra. This aspect of the energy evolution equations will therefore be deferred until these quantities are derived.

We now obtain expressions for the decorrelation rates ω_A, $\hat{\omega}_A$, γ_n, and γ_ϕ. From Eq. (36), $k^2\hat{\omega}_A$ can be extracted from Eq. (51) as the sum of all factors multiplying the correlation $\langle\hat{\psi}_k\hat{\phi}_{-k}\rangle$. However $k^2\hat{\omega}_A$ can also be extracted from Eq. (52) as the sum of factors multiplying $\langle\hat{\psi}_k\hat{n}_{-k}\rangle$. Consistency with Eq. (27) demands that only those factors that yield the same result contribute to the definition of $\hat{\omega}_A$. The remaining factors contribute to spectral transfer, but not as a coherent decay incorporated on the left hand side of Eqs. (36) and (37). Likewise, ω_A is extracted from Eq. (50) as the sum of factors multiplying the correlation $\langle\hat{\psi}_{-k}\hat{\phi}_k\rangle$ that also appear as factors multiplying $-\langle\hat{\psi}_{-k}\hat{n}_k\rangle$. Starting with the kinetic energy, Eq. (51), we obtain

$$\hat{\omega}_A k^2=\sum_{k'}\frac{4A^2_{k',k}[(k-k')^2-k'^2](k'^2-k^2)|\hat{\psi}_{k'}|^2}{\hat{\omega}_A''(k-k')^2}. \tag{60}$$

This expression is valid in both the long and short wavelength limits. It has the typical structure for decorrelation rates in EDQNM and other statistical closures, with the rate appearing recursively as a decorrelation within a spectrum sum on the right hand side of its defining equation. The nonlocality of the Alfvénic decorrelation allows the approximation $k'\ll k$, under which $\hat{\omega}_A''\approx\hat{\omega}_A$ and

$$(\hat{\omega}_A)^2\cong-\sum_{k'}(\mathbf{k}'\times\mathbf{z}\cdot\mathbf{k})^2|\hat{\psi}_{k'}|^2. \tag{61}$$

This expression is dimensionally equivalent to the heuristic treatment of the Alfvénic decorrelation introduced in the preceding section [e.g., in Eqs. (9)–(11)] but differs in three crucial ways. First, from Eq. (61), $\hat{\omega}_A$ is imaginary, indicating that $\hat{\omega}_A$ in fact describes a decorrelation process and is not a coherent oscillation. Second the fluctuation enters the right hand side of Eq. (61) as the absolute value squared, reflecting the well known fact that oppositely propagating Alfvénic disturbances are required for decorrelation. Finally, the spectrum sum indicates that the magnetic field on which these disturbances propagate is a superposition of Fourier components. Provided the spectral index for the magnetic energy is less than -1 this sum is dominated by the smallest wave numbers. The internal energy evolution equation also yields an expression for $-\hat{\omega}_A k^2$, obtained as the factor multiplying the correlation $\langle \hat{\psi}_k \hat{n}_{-k} \rangle$. The expression is identical to Eq. (60).

The Alfvénic decorrelation of the magnetic equation $-\hat{\omega}_A k^2$ is found from factors in Eq. (50) proportional to $\langle \hat{\phi}_k \hat{\psi}_{-k} \rangle$, which also appear as factors proportional to $\langle \hat{n}_k \hat{\psi}_{-k} \rangle$:

$$-\omega_A k^2 = \sum_{k'} 4A_{k',k}^2 k^2 \left\{ \frac{|\hat{\psi}_{k'}|^2}{\omega_A''} - \frac{\gamma_{\phi''} \langle \hat{\phi}_{k'} \hat{n}_{-k'} \rangle}{\hat{\omega}_A'' [\gamma_n'' + (k-k')^2 \gamma_{\phi''}]} \right\}. \tag{62}$$

From the expressions for γ_ϕ and γ_n derived below, the second term is negligible. Consequently $\omega_A^2 = \hat{\omega}_A^2$ and $\omega_A = \pm \hat{\omega}_A$. For the lower branch,

$$\omega_A = -\hat{\omega}_A = \hat{\omega}_A^*, \tag{63}$$

which is the form of the Alfvénic decorrelation in the three-wave interaction model of Eqs. (6)–(8) when $k_0 \ll k$. This symmetry of the basic equations dictates the choice of the lower branch.

The fluid straining decorrelation rates are obtained in a similar fashion. They apply to diagonal terms, or spectral energy densities, rather than cross correlations. The rate γ_ϕ is obtained from Eq. (51) as the factor multiplying the kinetic energy density $k^2 |\hat{\phi}_k|^2$,

$$k^2 \gamma_\phi = \sum_{k'} \frac{4A_{k',k}^2 [(k-k')^2 - k'^2]}{[\gamma_n'' + (k-k')^2 \gamma_\phi'']} \times \left\{ \frac{-(k'^2 - k^2) \gamma_n'' \langle \hat{\psi}_{k'} \hat{\phi}_{-k'} \rangle}{(k-k')^2 \hat{\omega}_A''} + (k'^2 - k^2) |\hat{\phi}_{k'}|^2 + \langle \hat{\phi}_{k'} \hat{n}_{-k'} \rangle \right\}. \tag{64}$$

The rate γ_n is obtained from Eq. (52) as the factor multiplying the internal energy density $|\hat{n}_k|^2$,

$$\gamma_n = \sum_{k'} \frac{4A_{k',k}^2 |\hat{\phi}_{k'}|^2}{[\gamma_n'' + (k-k')^2 \gamma_\phi'']}. \tag{65}$$

These equations must be evaluated jointly in either the long or short wavelength limit. For $k \ll 1$, $\gamma_\phi k^2 \ll \gamma_n$, and Eq. (65) yields

$$\gamma_n = \sum_{k'} \frac{4A_{k',k}^2 |\hat{\phi}_{k'}|^2}{\gamma_n''}, \tag{66}$$

with $\gamma_\phi \sim k^2 \gamma_n$. Equation (66) is a typical eddy damping expression in EDQNM. Because the nonlocal coupling of disparate-sized eddies corresponds to a distortion-free sweeping of small eddies by large eddies, the summation in Eq. (66) is limited to local triads with $k \sim k'$. Dimensionally, $\gamma_n \sim k^2 \hat{\phi}_k$, which is an eddy turnover rate. For $k \gg 1$, $k^2 \gamma_\phi \gg \gamma_n$, and Eq. (64) yields

$$k^2 \gamma_\phi = \sum_{k'} \frac{4A_{k',k}^2 [(k-k')^2 - k'^2](k'^2 - k^2) |\hat{\phi}_{k'}|^2}{(k-k')^2 \gamma_\phi''}, \tag{67}$$

with $\gamma_n \sim \gamma_\phi / k^2$. This expression is subject to the same triad restriction as that of Eq. (66), and leads to a decorrelation rate that is also an eddy turnover rate.

IV. SPECTRAL LAWS

From the scalings of the spectrum equations, Eqs. (50)–(51), we can obtain inertial range power laws for the spectral energy densities. Following Obukov, we solve simple asymptotic balances between scale-independent dissipation rates and the dominant spectral transfer rate of each equation. The dissipation rate is equal to an energy input rate created by external forcing at the outer scale. In accordance with the notion of spectral energy transfer that is local in wave number space we take $k \sim k'$. For ω_A, $\hat{\omega}_A$, γ_n, and γ_ϕ we use the values obtained in the preceding section, with the Alfvénic rates governed by magnetic fluctuations near the outer scale $k_0 \ll k$. Where equipartition is indicated by the nonlinear eigenmodes, we assume that the dissipation rate is equal in the two equations whose energies are in equipartition. The equipartition assumption is then verified *a posteriori* from the balances, which must yield identical power laws for the two energy densities. To determine which term dominates from the right hand side of each energy equation, we examine asymptotic limits for $k \to 0$ and $k \to \infty$. A balance with any selected term establishes power law dependencies for each energy, or equivalently, for each fluctuation amplitude. These dependencies can be substituted into the remaining terms. Only if the remaining terms vanish in the appropriate asymptotic limit, is the balance with the selected term a dominant balance.

We begin with the long wavelength limit, and verify the power laws previously predicted from a heuristic analysis like that of Sec. II.[1] We consider first the magnetic equation, Eq. (50), and note that with $\gamma_n \gg \gamma_\phi$, we must consider terms that go as $k^6 \hat{\psi}^3 \hat{\phi}$, $k^6 \hat{\psi} \hat{\phi}^3$, $k^6 \hat{\psi}^3 \hat{n}$, and $k^6 \hat{\psi} \hat{\phi}^2 \hat{n}$. We ignore the phase content of the correlations in Eq. (50) and deal only with amplitudes. For the Alfvénic eigenmode in the long wavelength regime [Eqs. (29)–(31)], $\hat{\psi}$ and $\hat{\phi}$ have equal amplitude, reducing the balances to two. The balance involving $\hat{\psi}$ and $\hat{\phi}$ is dominant, as will be apparent once n is determined from the internal energy equation. Therefore,

$$\epsilon = A_{k',k}^2 k^2 \frac{|\hat{\psi}_{k'}|^2 \langle \hat{\psi}_{k'} \hat{\phi}_{-k'} \rangle}{\omega_A''} \approx \frac{k B_k^3 V_k}{B_{k_0}}, \tag{68}$$

where $B_{k_0} = k_0 \hat{\psi}_{k_0} = V_A$. Under equipartition, $B_k \approx V_k$, yielding

$$E_M(k) = \frac{B_k^2}{k} = \frac{\epsilon^{1/2} B_{k_0}^{1/2}}{k^{3/2}}. \tag{69}$$

In the kinetic energy equation we must confront the possibility of competition between spectral transfer rates governed by the Alfvénic decorrelation (involving $G_{A\phi}$ and G_A) and those governed by the fluid straining decorrelation (involving $G_{\phi\phi}$ and G_ψ). The former go as $\hat{\psi}\hat{\phi}^3/\hat{\omega}_A$, while the latter go as $k^2\hat{\phi}^4/\gamma_n$. For equipartition of $\hat{\psi}$ and $\hat{\phi}$ the ratio of the former to the latter is $\gamma_n/k^2\hat{\omega}_A$. For the dominant balance of Eq. (68), $\gamma_n/\hat{\omega}_A \sim (k/k_0)^{-1/4}$, and is small in an inertial range. However, k^{-2} can be very large for a spectrum whose outer scale is 12 orders of magnitude larger than the ion gyroradius. Consequently, the Alfvénic terms dominate, leading to a balance

$$\epsilon = A^2_{k',k}[(k-k')^2-k'^2]\frac{(k'^2-k^2)}{(k-k')^2}\frac{|\hat{\phi}_{k'}|^2\langle\hat{\psi}_{k'}\hat{\phi}_{-k'}\rangle}{\omega''_A} \approx \frac{kV_k^3B_k}{B_{k_0}}, \tag{70}$$

and a spectrum $E_K(k)=E_M(k)$. The equipartition of the eigenmode is thus justified by the spectrum balances. The internal energy equation also has a competition between the Alfvénic and fluid straining decorrelation rates. Here terms whose decorrelation is governed by the former have an extra factor of k^2. Consequently the fluid straining terms dominate, leading to the balance

$$\epsilon_I = A^2_{k',k}\frac{|\hat{n}_{k'}|^2|\hat{\phi}_{k'}|^2}{\gamma''_n} \simeq \frac{k^2V_k^2n_k^2}{kV_k} = kV_kn_k^2. \tag{71}$$

The fluid straining terms with a single factor of $\hat{\phi}$ go as $k^2\hat{n}_k^3/V_k$, and are found to be subdominant when checked with the scaling of Eq. (71). The balance of Eq. (71) is that of a passive scalar. The spectrum is

$$E_I(k) = \frac{n_k^2}{k} = \frac{\epsilon_I}{\epsilon^{1/4}B_{k_0}^{1/4}k^{7/4}}. \tag{72}$$

The density decays more steeply than the other two spectra. Despite assertions to the contrary,[19] a passive scalar has the same spectral index as the advecting flow only for the special case of $-5/3$.

We consider now the short wavelength regime, starting with the magnetic equation. From the preceding section $\gamma_\phi \gg \gamma_n$ in this limit. All denominators of $G_{A\phi}$ and G_{An} go as $k^2\gamma''_\phi$, making all terms small except the first of the terms proportional to $\gamma''_\phi/\hat{\omega}_A$ in each of $G_{A\phi}$ and G_{An}. These go as $\hat{\psi}^3\hat{\phi}/\hat{\omega}_A$ and $\hat{\psi}^3\hat{n}/\hat{\omega}_A$. The four terms of G_A also have one or the other of these forms. The dominant term is $\hat{\psi}^3\hat{n}/\hat{\omega}_A$ because fewer factors of k are absorbed to convert $\hat{\psi}_k$ to B_k and $\hat{\phi}_k$ to V_k. The dominance can be verified *a posteriori* once $\hat{\psi}_k$, $\hat{\phi}_k$, and $\hat{n}_k$ are determined. Appealing to the Alfvénic eigenvector in the high k limit, we assume equipartition of $\hat{n}_k$ and B_k. Thus, from the dominant balance

$$\epsilon = A^2_{k',k}k^2\frac{|\hat{\psi}_k|^2\langle\hat{\psi}_{k''}\hat{n}_{-k''}\rangle}{\omega''_A} \approx \frac{k^2B_k^3n_k}{B_{k_0}} \simeq \frac{k^2B_k^4}{B_{k_0}}, \tag{73}$$

the spectrum is

$$E_M(k) = \frac{B_k^2}{k} = \frac{\epsilon^{1/2}B_{k_0}^{1/2}}{k^2}. \tag{74}$$

In passing from long to short wavelengths the magnetic spectrum steepens from $k^{-3/2}$ to k^{-2}. The assumption of equipartition can be verified by calculating the spectrum of $\hat{n}_k^2$ from the density equation. Repeating the arguments used for the magnetic equation, of the terms of Eq. (52) that decorrelate on the Alfvénic time scale, those that go as $\hat{\psi}^3\hat{n}/\hat{\omega}_A$ are largest. Of the terms that decorrelate on the fluid straining time scale, the first terms of G_ψ and G_ϕ are the largest. With its multiplicative factors, the former goes as $A^2k^2\hat{\phi}^2\hat{\psi}^2/\gamma$, while the latter go as $A^2V\hat{n}^2$. Once V_k is determined these terms will be found to be smaller than $A^2k^2\hat{\psi}^3\hat{n}/\hat{\omega}_A$. Hence the dominant balance is

$$\epsilon = A^2_{k',k}k^2\frac{(k'^2-k^2)}{(k-k')^2}\frac{|\hat{\psi}_{k'}|^2\langle\hat{\psi}_{k'}\hat{n}_{-k'}\rangle}{\hat{\omega}''_A}. \tag{75}$$

Assuming the partition of the Alfvénic eigenvector $k\hat{\psi}_k = \hat{n}_k$, this balance yields $E_I(k)=E_M(k)$, confirming the equipartition assumption. In Eq. (75), the density enters in a correlation with $\hat{\psi}_k$. As with all cross correlations there is a phase angle. However, from the Alfvénic eigenvector, the cosine of the angle is unity. Turning to the kinetic energy equation, we note that the terms in $G_{A\phi}$, $G_{\phi\phi}$, and G_ψ with a factor of (k'^2-k^2) or $[(k-k')^2-k'^2]$ are larger than the other terms. These terms go as $A^2k^2\hat{\psi}^3\hat{\phi}/\hat{\omega}_A$, $A^2k^2\hat{\phi}^4/\gamma_\phi$, and $A^2k^2\hat{\phi}^2\hat{\psi}^2/\gamma_\phi$. The second term dominates, yielding

$$\epsilon_K = A^2_{k',k}[(k-k')^2-k'^2]\frac{(k'^2-k^2)}{(k-k')^2}\frac{[\hat{\phi}_{k'}|^2|\phi_{k'}|^2]}{\gamma''_\phi} \simeq kV_k^3. \tag{76}$$

This is the balance of the Kolmogorov spectrum, from which

$$E_K(k) = \frac{\epsilon_K^{2/3}}{k^{5/3}}. \tag{77}$$

Under this dependence, it is readily verified that the subdominant terms are indeed smaller. There is clearly a dramatic change in the way in which kinetic energy is cascaded in the long and short wavelength limits. In the former it is cascaded principally through the Alfvénic coupling, undergoing a continuous conversion and reconversion process with the magnetic energy. In the short wavelength regime, self-advection of flow dominates, and kinetic energy is transferred between scales with negligible coupling to the magnetic field.

The spectral laws of the long wavelength regime cease to hold at a critical wave number that is less than unity. Returning to the discussion immediately following Eq. (69), the Alfvénic and fluid straining terms in the kinetic energy equation become equal when $\gamma_n/k^2\hat{\omega}_A=1$. Solving this expression for k yields the critical wave number $k_{\mathrm{crit}}=L^{-1/9}$, where L is the outer scale normalized to the ion gyroradius. If the outer scale is 12 decades removed from the scale of the gyroradius, as in the ISM, the critical wave number is $k_{\mathrm{crit}} = 10^{-4/3}$. At this wave number fluid straining takes over in the kinetic energy equation. Because γ_n remains smaller than γ_ϕ the fluid straining term in the energy balance has an additional factor of k^2 relative to the balance at $k \gg 1$. This produces a steep decay of kinetic energy with $E_K(k) \sim k^{-3}$. Provided the fluid straining term of the internal energy equation remains dominant, the internal energy decay levels off until the Alfvénic term becomes important. This behavior allows the internal energy to become larger than the kinetic energy. Once this happens, the magnetic energy drops more sharply, allowing equipartition with the internal energy. The physics of this transition subrange is clearly complicated, and many terms from Eqs. (50)–(52) potentially come into play. Analysis of this subrange will be undertaken in a future publication. The wave number at which the $k \gg 1$ spectral laws begin to emerge depends on the details of the transition subrange.

Given the complexities of the nonlinear inertial interactions probed in this paper, we have not included dissipation in the analysis. Yet for applications such as interstellar turbulence it is worth considering at what wave number dissipation might occur relative to other features such as those discussed above. Recently, Leamon *et al.* have inferred the threshold wave number of the dissipation range in interplanetary turbulence at one AU.[20] The issue has its own complexities, particularly since the apparent isotropy of the dissipation range strongly suggests kinetic processes, including cyclotron resonance damping, ion and electron Landau damping, and kinetic Alfvén wave excitation. The onset of the dissipation range is seen as a break in the spectral index of the *frequency* spectrum from a value -1.67 to -2.91. The corresponding wave number is placed between $k=0.2$ and 0.3. The critical wave number derived in the preceding paragraph for the transition from Alfvénic to fluid straining decorrelation in the kinetic energy equation falls at a somewhat lower wave number of $k_{\text{crit}}=0.04$. Consequently it is possible that the inertial (nonlinear) kinetic Alfvén wave physics presented in this paper enters the cascade before dissipation. However, these critical wave numbers are sufficiently close that, given uncertainties, both processes may be involved in the interstellar index of -1.9, which after all, is much less steep than the interplanetary index of -2.9. Clearly the role of dissipation in nonlinear interactions near the inner scale is an important issue for future consideration.

The spectral indices derived above are meaningful only if there is a sufficiently large spectral subrange in which the balances can become dominant. However, even in the absence of such a subrange, the asymptotic balances indicate which interactions tend to dominate, and the approximate outcome in terms of energy partitions. The analysis assumes appropriate forcing in all three equations, but it can be used to infer what happens when a single field is forced, as is done in the simulations described in the next section. For example, if the magnetic fluctuation is the only field that is externally forced, it can be expected that the field to which it couples Alfvénically will be nonlinearly excited, and its energy will be brought into equipartition with the magnetic energy. Thus in the long wavelength regime, the kinetic energy will be forced into an equipartition with the magnetic energy. Without external forcing or an advective source from an equilibrium gradient, the density will be excited only through the Alfvénic coupling of G_{An}. This coupling is weak for $k \ll 1$, while the turbulent advection, which depletes internal energy by cascading it to dissipated scales, is strong. Consequently, the internal energy will be small compared to the magnetic and kinetic energies. In the short wavelength regime the roles of kinetic and internal energy reverse. The Alfvénic coupling of n and B through G_{An} and G_A now dominates the advection of density, and the internal energy is nonlinearly excited to equipartition with the magnetic field. For kinetic energy the Alfvénic coupling through $G_{A\phi}$ and G_A is now weaker than self-advection, which acts to deplete kinetic energy by cascading it to dissipated scales. Hence, the kinetic energy falls to a level that is below that of magnetic and internal energies. The decoupling of kinetic energy and the strong interaction of magnetic field and density through kinetic Alfvén disturbances, is precisely the situation described by the CEMT model of earlier studies.[16,21]

V. NUMERICAL ILLUSTRATIONS

The general behavior described in the preceding sections is evident from numerical results using a spectral code to solve the basic dynamical equations. The same code was used in Ref. 7 to measure the turbulent response, or characteristic decorrelation rates, associated with nonlinearities of the model. Details of the code are given in that reference. From those studies, the basic energy balances of the long wavelength regime, Eqs. (68), (70), and (71), were verified numerically. The simulations did not have sufficient spatial resolution to measure spectral indices. Even in high resolution simulations it is difficult to distinguish between the indices that differ only slightly, e.g., $k^{-3/2}$ vs $k^{-5/3}$. However, from large ensembles tracking the temporal response to perturbations, it was possible to clearly distinguish the fluid straining rate in the denominator of Eq. (71) from the Alfvénic rates in the denominators of Eqs. (68) and (70). These methods established that the Alfvénic and fluid straining decorrelation rates are given, to good accuracy, by Eqs. (61) and (66), and that the characteristic frequencies of responses are based on the decorrelation rates according to Eqs. (27) and (28). Moreover, these studies verified the structure of the response function eigenvectors, Eqs. (29)–(34).

Here we examine the energy transfer dynamics and partitions that characterize long and short wavelength limits as illustrations of the physics presented in earlier sections. In the numerical solutions the magnetic field was the only fluctuation with forcing, creating the situation described at the end of the preceding section where the kinetic and internal energies are excited solely by coupling across fields. The forcing was accomplished through a linear instability drive introduced into the magnetic energy equation. Unstable modes were typically restricted to a single wave number at long wavelength. Hyper dissipative damping was used in all three fields to dissipate energy at the smallest scales. The solutions were run until the instability was saturated by nonlinear spectral transfer, and a steady state was reached. At this point an energy transfer diagnostic was turned on. The energy transfer diagnostic records the rate of energy transfer

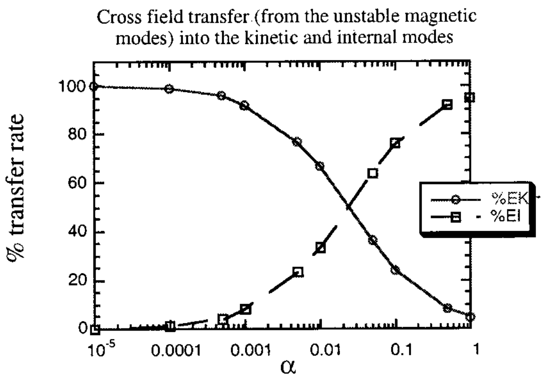

FIG. 2. Percent of energy transfer from the magnetic equation going to fluctuations of flow (circles) and density (squares) as a function of α. The parameter α sweeps wavelength regimes, with $\alpha \ll 1$ representing a spectrum in the long wavelength regime and $\alpha = 1$ representing a short wavelength spectrum.

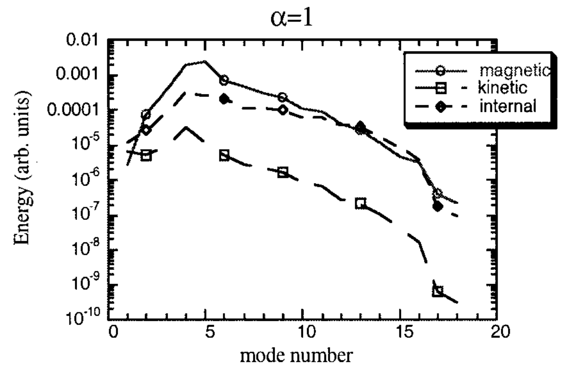

FIG. 4. Wave number spectrum for turbulence in which all scales lie in the short wavelength regime ($\alpha = 1$). Kinetic energy is much smaller than magnetic and internal energies, which are equipartitioned.

into and out of annular bands in the two-dimensional space of k_x and k_y for the three forms of energy (magnetic, kinetic, and internal). The procedure was repeated for spectral ranges lying entirely within long and short wavelength subranges, or in the transition region between the subranges. The sampling of these subranges was varied continuously with the parameter α defined from the normalization used in Ref. 12 for the basic model equations. The parameter α is the wave number squared of the longest wavelength mode normalized to the ion gyroradius squared. For $\alpha = 10^{-5}$, the smallest value used in the computations, all modes were well within the long wavelength subrange. For $\alpha = 0.1$, the longest wavelength modes were marginally in the long wavelength regime, while shorter wavelength modes had $k > 1$. For $\alpha = 1$ the entire spectrum had $k > 1$.

Figure 2 shows the energy transferred from the magnetic equation, broken down into the percentage transferred to the kinetic and internal energy equations, as a function of α. Below $\alpha = 10^{-3}$ nearly all of the energy transferred by the magnetic energy equation passes to the kinetic energy equation. This indicates that magnetic energy cannot pass from one scale to another without being converted to another form of energy. The favored form is kinetic energy, in accordance with the Alfvénic interaction at long wavelengths. Around $\alpha = 0.01$ there is the beginning of significant energy transfer to the internal energy equation; at $k = 1$, nearly all of the energy is transferred to the internal energy equation and almost none is transferred to the kinetic energy equation. This figure clearly shows how the kinetic and internal energies reverse roles in going from long to short wavelengths, as predicted from the results of the spectrum analysis. Figures 3–5 show steady state spectra for three values of α corresponding to regimes with long, short, and intermediate wavelengths. In Fig. 3 $\alpha = 10^{-5}$, and there is an equipartition of magnetic and kinetic energies away from the unstable magnetic mode at $k = 5$. The internal energy is orders of magnitude lower for all wave numbers. In contrast, Fig. 4 shows the situation for $\alpha = 1$ where the magnetic and internal energies reach equipartition above $k = 5$. The kinetic energy is two orders of magnitude smaller. Figure 5 shows the intermediate case of $\alpha = 0.01$. The spectrum is clearly in a transition subrange, and all three fields are in rough equipartition. These results clearly illustrate the behavior predicted in the prior sections: in the long wavelength regime the magnetic field and flow strongly couple through the Alfvénic interaction, and density is a passive advectant; in the short wavelength regime the magnetic field and density strongly

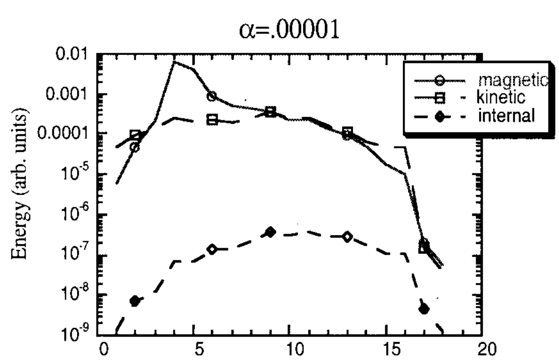

FIG. 3. Wave number spectrum for turbulence in which all scales lie in the long wavelength regime ($\alpha = 10^{-5}$). The system is forced by a linear instability in the magnetic equation with $k = 5$. Internal energy is much smaller than magnetic and kinetic energies, which are equipartitioned.

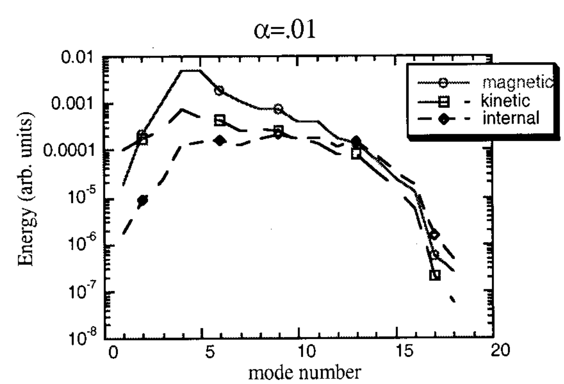

FIG. 5. Wave number spectrum for turbulence in an intermediate regime between long and short wavelengths ($\alpha = 0.01$). All three energies are comparable.

interact through the kinetic Alfvén wave, and flow becomes decoupled. The numerical results have not verified the values derived for spectral indices, but they have verified the physics and balances that uniquely lead to those values.

VI. CONCLUSIONS

We have considered here anisotropic magnetic turbulence in a model that reproduces in the long wavelength limit the salient features of turbulence in the warm diffuse ionized component of the interstellar medium. As the wavelength approaches the ion gyroradius the electron density goes from passive to active. To understand the dynamics associated with the coupling of electron density fluctuations to magnetic turbulence we have investigated both long and short wavelength regimes. While the asymptotic short wavelength regime is below the inner scale of the observed spectrum of interstellar turbulence, the physics of an active density is expected to play a role near the inner scale. We caution that other effects not considered here, such as dissipation, may enter at these scales, affecting the direct applicability of the present results to interstellar turbulence. The model is based on reduced MHD with electron density fluctuations incorporated through the electron pressure in Ohm's law and parallel compression of electron density in the electron continuity equation. We have taken the limit of strong anisotropy, in which the wave number along the local magnetic field is negligible.

On the basis of a systematic statistical closure of the energy evolution equations, we find that even for strong anisotropy, Alfvénic decorrelation *alone* mediates energy transfer in the equation for the magnetic field. Alfvénic interactions couple magnetic field to the flow at long wavelengths and magnetic field to density at short wavelengths. The coupling, which drives equipartition between the coupled fields, decorrelates on the Alfvén time scale. The fluid straining decorrelation, or eddy turnover rate, affects only the cascade of internal energy at long wavelengths, and the cascade of kinetic energy at short wavelengths. The spectra associated with these processes have distinct features in the two regimes. For long wavelengths the spectral indices are $-3/2$ for magnetic and kinetic energy, and $-7/4$ for internal energy. For short wavelengths the spectral indices are -2 for magnetic and internal energy, and $-5/3$ for kinetic energy. The critical wave number at which the long wavelength spectra cease to hold is $k_{\rm crit}=L^{-1/9}$, where L is the outer scale.

The statistical closure theory sorts out competing decorrelation processes whose relative importance and effects change over the spectrum. In general, Alfvénic and fluid straining interactions decorrelate on their respective time scales only, with the interactions competing to govern the spectral transfer rate. To accurately capture the physics, the closure must properly account for the nonlocal (in wave number space) Alfvénic interactions and their off-diagonal coupling character. The expression provided by the closure for the Alfvénic decorrelation rate shows that it is truly a decorrelation, despite an underlying motion consisting of propagating waves. The expression also reveals its nonlocal character, and the necessity of oppositely propagating disturbances. Magnetic energy is seen to cascade from one scale to another through conversion to kinetic or internal energy and reconversion back to magnetic energy. This process is governed by the cross correlations $\langle\hat{\psi}\hat{\phi}\rangle$ and $\langle\hat{\psi}\hat{n}\rangle$. The relative phase of these correlations is set by the nonlocal Alfvénic interaction, which has the character of a shear Alfvén wave in the long wavelength regime, and a kinetic Alfvén wave in the short wavelength regime. Fluid straining interactions are seen to govern the cascade of internal energy at long wavelengths, and kinetic energy at short wavelengths. The closure equations also contain a complicated transition region around $k=1$. The rich and complicated interplay between Alfvénic and fluid straining processes revealed by consideration of a spectrum containing both long and short wavelength limits dispels the overly simple notion that a single decorrelation rate applies to all interactions. Moreover, the energy partition structure directly reflects the role of Alfvénic versus fluid straining motions in the nonlinear energy transfer. This suggests that the observed Alfvénic equipartition of flow and magnetic field fluctuations in simulations is incompatible with a decorrelation that is not Alfvénic.

This study indicates that the spectral laws of the long wavelength subrange do not extend all the way to $k=1$. Thus, provided dissipation does not cut off the spectrum of interstellar turbulence at the inner scale, the smallest scales likely reflect a transition to the regime of active density. Because there is a transition subrange before the short wavelength spectral laws come into play, it is not clear if the measured index of -1.9 for angular broadening of extragalactic radio sources reflects the short wavelength index or other physical processes such as dissipation. A magnetic energy spectrum that is steeper than the MHD result of $k^{-3/2}$ is observed in MST, which also makes the steeper spectrum of k^{-2} derived herein appealing. However, in laboratory plasmas the spectrum is limited in extent, covering only a few decades. In this situation the gradient of mean density plays a greater role than that likely for interstellar turbulence. Therefore, consideration of the role of the diamagnetic frequency ω_* should be undertaken for more detailed comparison with MST.

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