Nonlinear inward particle flux component in trapped electron mode turbulence

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Trapped electron turbulence is shown to have a significant inward particle flux component associated with nonlinear deviations of the density-potential cross correlation from the quasilinear value. The cross correlation is altered because the density advection nonlinearity mixes a linearly stable eigenmode with the eigenmode of the instability. The full nonlinear flux is evaluated by solving spectrum balance equations in a complete basis spanning the fluctuation space. An ordered expansion for small collisionality, perpendicular wave number, and temperature/density-gradient instability threshold parameter enables an analytic solution for a weakly driven regime. The solution quantifies the role of zonal modes on transport via their saturation of the turbulence under intensely anisotropic transfer. The inward transport is neither diffusive nor convective, but is driven by temperature gradient and enhanced by flat density gradients. It is slightly smaller than the outwardly directed flux associated with the growing eigenmode, making the flux a small fraction of the quasilinear value. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2212403]

I. INTRODUCTION

It has long been recognized that the centrally peaked density profiles of toroidal fusion devices frequently require some form of inward particle motion. Neoclassical theory provides such motion in the form of the Ware pinch,[1] but it is often too weak to reproduce observed profiles, given known particle sources. Consequently, anomalous pinch mechanisms are believed to operate in fusion plasmas. Anomalous pinches are driven by turbulence and, in principle, are much larger than collision-driven fluxes. The first anomalous pinch was postulated to arise for collisional free-streaming, nonadiabatic electrons subject to a thermal (temperature gradient) force in a fluctuation driven through ion dynamics.[2] Subsequently this type of mixing mode mechanism was shown to apply to collisionless or weakly collisional trapped electrons below a collisionality threshold.[3] These so-called thermodiffusive pinches can dominate outward diffusive flux components to produce a net inward particle flux. A second type of anomalous pinch labeled turbulent equipartition is associated with the adiabatic invariance of trapped electron motion and is effectively driven by magnetic field curvature, or the gradient of the safety factor.[4,5] These two types of fluxes emerge from a comprehensive kinetic treatment of the linear density response to the electrostatic potential in toroidal geometry. The density response includes both trapped and passing particles,[6–8] and thereby captures pinches associated with both.[9] The result is a unified quasilinear particle flux incorporating the known anomalous pinches driven by temperature and safety factor gradients.[10] The two types of pinches are also unified in a quasilinear formulation of the entropy production rate, ensuring consistency with thermodynamic constraints.[11] Recently there has been an active effort to corroborate the theoretical pinches and their conceptual underpinnings by comparisons with numerical models and experiments.[11–14] This body of work, which also has sought to determine the relative importance of different pinch mechanisms in various fusion experiments, has generally found that turbulent equipartition dominates in a number of present day devices where collisionality is not particularly low. However, thermodiffusion, which has a pinch at low collisionality, is expected to play a role in the International Thermonuclear Experimental Reactor (ITER).[15]

Research on pinches has relied heavily on quasilinear theory. Analytical calculations use the quasilinear approximation almost exclusively, but it has also been the framework by which experiments and numerical models are interpreted.[16] The quasilinear approximation uses a linear density response in calculating the particle flux, thereby missing the contribution of nonlinearity to the cross correlation between the density and the potential. Transport cross correlations have been found to be strongly nonlinear in turbulence models for trapped electron mode turbulence,[17] ion drift wave turbulence,[18] and Rayleigh–Taylor turbulence.[19] To be more concrete, the particle flux is given by

$$\Gamma = -\frac{c}{B_0}\sum_k k_y \, \mathrm{Im}\langle n_k \phi_{-k}\rangle . \tag{1}$$

The flux itself is nonlinear, of course, because it depends on a quadratic correlation. However, the simplest and most widely exploited method for evaluating the correlation is to use a linear eigenvector to express n_k as a linear response to the potential, $n_k = R_1(k)\phi_k$. Substitution of this expression into Eq. (1) yields the quasilinear flux. For simple drift waves the eigenvector of the unstable eigenmode produces outward transport.[20] The eigenvector of *stable* eigenmodes yields inward transport. Stable eigenmodes are generally assumed to make no contribution to transport after initial decay, because their amplitude ϕ_k is presumed to become zero under the linear damping. The quasilinear flux thus represents the steady-state particle transport driven by the turbulence-causing instability. Note that the response $R_1(k)$ follows from inverting a linearized equation for electron density evolution. The response is a function solely of k because the frequency is equated to an eigenfrequency, $\omega=\omega_k+\gamma_k$. This step makes the response a linear eigenvector, specifically the eigenvector of the linear instability, assuming ω is the eigenfrequency of the linear instability. In calculating the particle flux, the effect of a saturated instability can be captured if the fluctuation level $|\phi_k|^2$ is taken to be that of the saturated steady state. However, the assumption that the cross correlation continues to be governed by a linear response $R_1(k)$, even in saturation, is severe.

The effect of nonlinearity on the transport cross correlation is a difficult problem that has received only limited attention.[21] Analytically, the problem amounts to the inversion of a nonlinear operator whose form depends on the solution of a nonlinear eigenmode problem. Even if the evolution equation for the density-potential correlation is formulated in a closure,[22] its solution requires a temporal inversion that remains highly nontrivial because of the way characteristic temporal responses are mixed nonlinearly. As a result, there has been no general procedure capable of evaluating the particle flux outside the quasilinear approximation. The nonlinear cross correlation is treated in numerical solutions of nonlinear models with particle transport physics, but, excepting particular cases,[7] is generally not investigated in relation to the quasilinear approximation. Therefore, there are fundamental unanswered questions about particle transport. For example, when does the quasilinear approximation break down? What form does the particle flux take when this approximation breaks down? What underlying nonlinear effects are at work and how do they operate? How can they be represented and described? What are their effects on the anomalous pinch?

Recent work offers a new approach for determining nonlinear phase correlations and nonlinear eigenmodes.[17] This approach provides a systematic treatment of frequencies in strong turbulence where linear frequencies are mixed in an amplitude-dependent fashion. It removes the uncertainties and ad hoc assumptions in prior work.[21] The approach expands the density, and any other fluctuation present in the model, in a complete basis set of linear eigenmodes. The density is expanded according to

$$n_k(t)=R_1(k)\beta_1(k,t)+R_2(k)\beta_2(k,t)+\cdots, \tag{2}$$

where $\beta_j(k,t)$ are nonlinearly evolving amplitudes of a complete basis set of linear eigenmodes and $R_j(k)$ are eigenvector components in the original function space of the density and the other fluctuations of the model. If the fluctuations in the turbulent state project onto one and only one eigenmode, that of the instability, the quasilinear flux is recovered. Conversely, because the basis set is complete, any deviation from the quasilinear flux necessarily implies the excitation of other members of the basis set. The other members correspond to different instabilities with weaker drive, and distinct, neutrally stable and damped eigenmodes of the plasma dielectric. Equation (2), and comparable expansions for the other fluctuations, are referred to as the eigenmode decomposition.

This approach has been implemented in a fluid model for trapped electron mode (TEM) turbulence.[17] In that system the unstable TEM eigenmode is mixed with a second eigenmode that is stable for all wave numbers. In the weakly collisional regime the latter is excited to a stationary, finite-amplitude level by the nonlinearity of electron density advection. The resulting nonlinear eigenmode is an amplitude-dependent combination of the unstable eigenmode, which contributes an outward component to the particle flux, and the stable eigenmode, which contributes an inward component. This model is restricted solely to the dynamics of TEM. Unlike the comprehensive quasilinear thermodiffusive pinches that incorporate the physics of ion temperature gradient (ITG) turbulence and nonadiabatic electrons, there is no ion temperature gradient drive. Moreover, the electron free energy source in TEM involves both electron density and temperature, i.e., both are destabilizing. Consequently, thermodynamics constrains the net flux to be positive. This means that inward flux contributions arising from the nonlinear excitation of the damped eigenmode are smaller than the outward quasilinear flux associated with the unstable eigenmode. However, inward and outward contributions are similar in magnitude, so there is a significant reduction of the net outward flux. Moreover, the inward and outward components have different scalings with density and temperature gradients, as shown in this paper. We choose the TEM model because its simplicity relative to mixing-mode models allows a complete analytical solution of the nonlinear flux and thereby provides specific answers to the questions posed above. Because the premise of the mixing mode is that the electron dynamics act weakly on the ion dynamics,[2,3] it is possible that the nonlinear physics described herein makes the net inward flux of the quasilinear thermodiffusive pinch even more strongly inward in regimes of ITG instability. However, that is something that must be shown from the details of the saturated state.

We compute here the particle flux associated with the TEM fluid model in the weakly collisional regime. The contributions of the unstable and stable eigenmodes are calculated using the eigenmode decomposition. Expansion of $\langle n_k\phi_k^*\rangle$ in the eigenmode decomposition yields a flux that depends on auto correlations $|\beta_1(k)|^2$, $|\beta_2(k)|^2$, and cross correlations $\mathrm{Re}\langle\beta_1^*(k)\beta_2(k)\rangle$ and $\mathrm{Im}\langle\beta_1^*(k)\beta_2(k)\rangle$. Consequently the stationary values of these quantities in saturation must be evaluated. Saturation is dual in nature. Energy moves from the unstable to the stable manifold. (The manifolds are k_x-k_y planes for each eigenmode.) Energy also moves from wave numbers with $k_y\neq 0$ to zonal wave numbers with $k_y=0$.[23] The zonal modes dominate the spectrum, but are excluded from the flux expression because $k_y=0$ makes the zonal component vanish. Nevertheless the anisotropic spectral energy transfer that populates the zonal modes is so important to saturation that it strongly affects the levels of the eigenmode auto and cross correlations.[23] This problem thus provides an illustration of the crucial distinction between zonal modes, which do not contribute directly to the particle flux, but strongly influence saturation levels, and damped eigenmodes, which not only contribute to the flux, but are responsible for the inward component and the deviation from quasilinear values.

To determine whether the flux is diffusive, convective, or of some other form, we must know its dependence on the gradients of density and temperature. This requires us to find the gradient scalings of saturated amplitudes. Previous work determined only the scaling of saturated amplitudes on the ratio of collision frequency to diamagnetic frequency. This ratio is a function of density gradient, but there is also density and temperature gradient dependence in η_e, the density to temperature gradient scale length ratio. In this paper we tackle the difficult and previously untreated problem of resolving the η_e dependence in the saturated amplitudes.

The remainder of this paper is organized as follows. In Sec. II the basic model is introduced along with the eigenmode decomposition in the context of the particle flux. Section III describes the calculation of saturation levels to determine the scaling of the flux on driving gradients near the instability threshold. Discussion and conclusions are offered in Sec. IV.

II. FLUX EIGENMODE DECOMPOSITION

The fluid model for TEM turbulence is given by

$$\frac{\partial n_k}{\partial t} + \nu n_k + [ik_y v_D \hat{\alpha} - \nu]\phi_k = b_n(k) \equiv -\sum_{k'} (\mathbf{k}' \times \mathbf{z} \cdot \mathbf{k})\phi_{k'} n_{k-k'}, \tag{3}$$

$$\frac{\partial \phi_k}{\partial t} - \frac{\epsilon^{1/2}\nu}{1+k^2-\epsilon^{1/2}} n_k + \frac{[ik_y v_D(1-\hat{\alpha}\epsilon^{1/2}) + \nu\epsilon^{1/2}]}{1+k^2-\epsilon^{1/2}}\phi_k = b_\phi(k) \equiv -\sum_{k'} \frac{(\mathbf{k}' \times \mathbf{z} \cdot \mathbf{k})(k-k')^2}{1+k^2-\epsilon^{1/2}}\phi_{k'}\phi_{k-k'}, \tag{4}$$

where $n_k = \epsilon^{1/2} n_e + \phi_k$ is an effective density, n_e is the density of trapped electrons, ϕ_k is the potential, $\epsilon^{1/2}$ is the trapping fraction, ν is the detrapping rate, v_D is the diamagnetic drift velocity, $\hat{\alpha} = 1 + 3\eta_e/2$, and η_e is the ratio of gradient scale lengths for the density and temperature. A derivation of this model and the dimensionless normalizations for n, ϕ, t, x, and y are given in Ref. 17. The nonlinearities are advection of turbulent electron density, abbreviated as $b_n(k)$, and advection of vorticity, abbreviated as $b_\phi(k)$. Density advection has two spatial derivatives, vorticity advection has four. Consequently electron density advection is the dominant nonlinearity at large scales where $k \ll \sqrt{n/\phi}|_{\mathrm{rms}}$. We will concentrate on the long wavelength regime and consider only $b_n(k)$ in the subsequent analysis. Deviations from the quasilinear flux are most pronounced in this regime. Moreover, the particle flux is dominated by fluctuations with long wavelengths.

The eigenmode decomposition expresses the density and potential as combinations of the two linear eigenmodes, but with amplitudes $\beta_1(k,t)$ and $\beta_2(k,t)$ that evolve nonlinearly

$$\begin{pmatrix} n_k(t) \\ \phi_k(t) \end{pmatrix} = \beta_1(k,t)\begin{pmatrix} R_1 \\ 1 \end{pmatrix} + \beta_2(k,t)\begin{pmatrix} R_2 \\ 1 \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \beta_1(k,t) \\ \beta_2(k,t) \end{pmatrix} \equiv \mathbf{M}\begin{pmatrix} \beta_1(k,t) \\ \beta_2(k,t) \end{pmatrix}, \tag{5}$$

where $[R_1,1]$ and $[R_2,1]$ are the eigenvectors and $R_j(k)$ are the ratio n_k/ϕ_k for each eigenfrequency ω_j. The eigenvector components $R_j(k)$ are obtained by linearizing Eq. (4), replacing $\partial/\partial t$ with $-i\omega_j$, and solving for n_k. The result is

$$R_j(k) = -\frac{1+k^2-\epsilon^{1/2}}{\nu\epsilon^{1/2}}\left[i\omega_j - \frac{ik_y v_D(1-\hat{\alpha}\epsilon^{1/2}) + \nu\epsilon^{1/2}}{1+k^2-\epsilon^{1/2}} \right], \tag{6}$$

where the eigenfrequencies ω_j are the roots of the characteristic equation,

$$\omega^2(1+k^2-\epsilon^{1/2}) + \omega[-v_D k_y(1-\hat{\alpha}\epsilon^{1/2}) + i\nu(1+k^2)] - ik_y v_D \nu = 0. \tag{7}$$

Expressions for these frequencies are given in Ref. 17. The eigenvector components are specified only up to an overall amplitude. This permits us to take the second component of each eigenvector as unity. The overall amplitudes $\beta_1(k,t)$ and $\beta_2(k,t)$ are then governed by the nonlinear evolution through Eqs. (3) and (4).

The flux, Eq. (1), is expressed in the eigenmode decomposition as

$$\Gamma = -\sum_k k_y[\mathrm{Im}\, R_1|\beta_1|^2 + \mathrm{Im}\, R_2|\beta_2|^2 + \mathrm{Im}(R_1+R_2)\mathrm{Re}\langle\beta_1^*\beta_2\rangle + \mathrm{Re}(R_2-R_1)\mathrm{Im}\langle\beta_1^*\beta_2\rangle], \tag{8}$$

where Γ has been normalized to the product of sound speed and mean density, and all quantities are understood to be functions of wave number k. From Eq. (6) for R_j and the roots of the characteristic equation, the flux is

$$\Gamma = \sum_k k_y \left\{ \frac{[\hat{\alpha}(1+k^2)-1][1+k^2-\epsilon^{1/2}]}{(1-\hat{\alpha}\epsilon^{1/2})^3}\left(\frac{\nu}{k_y v_D}\right)|\beta_1|^2 - \frac{(1-\hat{\alpha}\epsilon^{1/2})}{\epsilon^{1/2}}\left(\frac{k_y v_D}{\nu}\right)|\beta_2|^2 - \frac{(1-\hat{\alpha}\epsilon^{1/2})}{\epsilon^{1/2}}\left(\frac{k_y v_D}{\nu}\right)\mathrm{Re}\langle\beta_1^*\beta_2\rangle - \frac{[2\epsilon^{1/2}-(1+k^2)(1+\hat{\alpha}\epsilon^{1/2})]}{(1-\hat{\alpha}\epsilon^{1/2})\epsilon^{1/2}}\mathrm{Im}\langle\beta_1^*\beta_2\rangle \right\}. \tag{9}$$

The first term of Eq. (9) is the quasilinear flux, while the remaining three terms are the nonlinear components of the flux, and lie wholly outside quasilinear theory. (In quasilinear theory the amplitude of the unstable eigenmode $|\beta_1|^2$ is identical to the fluctuation level $|\phi_k|^2$, and $\beta_2 = 0$.) As expected for an unstable eigenmode of a simple TEM model, the quasilinear flux is outward. The second term is a contribution coming entirely from the stable eigenmode, and, as expected, it is inward. One of the difficult aspects of Eq. (9) is that while the first two terms have definite signs rooted in simple physics, the signs of the third and fourth terms are not easily determined or tied in any simple way to transparent physical considerations. The third and fourth terms are cross correlations between the two eigenmodes.

As long as one is dealing with quasilinear theory, the sign of the flux is found from linear analysis; the fluctuation level can be supplied, if not from an analytic calculation, then from a simple estimate like a mixing length or from a level measured in experiment. For the nonlinear flux, however, it is necessary to know how the fluctuation level partitions into stable and unstable eigenmode components. Moreover, it is insufficient to know the squared amplitude of each eigenmode. The complex cross correlation must also be known. Mixing length estimates are of no use in this situation; $|\beta_1|^2$, $|\beta_2|^2$, $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$, and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$ must be solved from a saturation balance that properly accounts for excitation of the damped eigenmode and the nonlinear spectral energy distribution between all of these quantities. Without recourse to mixing length arguments, it is also impossible to determine how each term of the flux scales with driving gradients. This scaling indicates whether components are diffusive, convective, or of some other form. There is, of course, the partial scaling associated with the coefficients of the turbulent coupling. However, the levels also have scaling with the driving gradients, which likewise must be obtained from solution of the saturation balances. This task will be described in the next section.

The sign of Eq. (9) appears to be subject to a competition between terms, some of which have signs that are not readily evident. The flux, however, is nonnegative in the steady state. This follows from an exact constraint derived from the fluctuation energy,

$$W=\sum_k E(k)=\sum_k\left[(1+k^2-\epsilon^{1/2})|\phi_k|^2+\epsilon^{1/2}|n_k|^2\right]. \tag{10}$$

If we take the time derivative of Eq. (10), and use Eqs. (3) and (4) to replace $\partial n_k/\partial t$ and $\partial\phi_k/\partial t$ by the remaining terms governing evolution, the time derivative of the energy can be written

$$\frac{dW}{dt}=2\sum_k\left[-k_y v_D\hat{\alpha}\epsilon^{1/2}\,\mathrm{Im}\langle n_k\phi_k^*\rangle-\nu\epsilon^{1/2}|(n_k-\phi_k)|^2\right]. \tag{11}$$

This fairly simple form includes only the dissipative terms of the evolution equations, Eqs. (3) and (4), because the nonlinear transfer terms exactly cancel in the sum over k. The cancellation is an expression of energy conservation. The first term on the right hand side of Eq. (11) is proportional to the flux, hence Eq. (11) can be reexpressed as

$$\Gamma=\frac{1}{2v_D\hat{\alpha}\epsilon^{1/2}}\frac{dW}{dt}+\frac{\nu}{v_D\hat{\alpha}}\sum_k|n_k-\phi_k|^2. \tag{12}$$

This is the fluctuation-dissipation theorem for the TEM system [Eqs. (3) and (4)]. Its importance here is that if there is a steady state with $dW/dt=0$, $\Gamma\geq 0$, i.e., the flux is constrained to be nonnegative. This constraint is related, at least informally, to the second law of thermodynamics, with the conclusion that the flux cannot push particles up the density gradient unless energy is being depleted from the fluctuation spectrum $(dW/dt<0)$. Since the quasilinear part of the flux is positive when the eigenmode is unstable, the nonlinear components, if negative, must be smaller than the quasilinear component. It should be cautioned, however, that saturated states can be very nonstationary,[18,24] with persistent long-time-scale oscillations during which dW/dt and Γ are negative. Moreover, the constraint applies to the simple TEM model of this paper and not to mixing-mode dynamics where ion free energy can drive inward particle transport in steady state,[2,3] or to the comprehensive models like those of Refs. 6–8. Equation (12) is an exact integral constraint. It applies as a sum over k space, but not pointwise in k space. If, as is commonly done, the flux is approximated from pointwise energy balances in the spectrum subrange of the instability (mixing length is an example), the approximation may not satisfy Eq. (12).

III. THRESHOLD SCALING OF SATURATED STATE

We now take up the matter of evaluating the flux, Eq. (9), from an analytic solution giving the quantities $|\beta_1|^2$, $|\beta_2|^2$, $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$, and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$. These quantities are ultimately governed by the evolution equations for the eigenmode amplitudes. Recasting Eqs. (3) and (4) in the eigenmode decomposition yields the desired evolution equations,

$$\begin{pmatrix}\dot{\beta}_1(k)\\ \dot{\beta}_2(k)\end{pmatrix}+\begin{pmatrix}i\omega_1 & 0\\ 0 & i\omega_2\end{pmatrix}\begin{pmatrix}\beta_1(k)\\ \beta_2(k)\end{pmatrix}=\frac{1}{R_1(k)-R_2(k)}\begin{pmatrix}b_n\\ -b_n\end{pmatrix}, \tag{13}$$

where b_n is understood to be evaluated using the substitution $n_k=R_1\beta_1+R_2\beta_2$ and $\phi_k=\beta_1+\beta_2$. It is possible to simplify Eq. (13) somewhat, because numerical solutions show that $\beta_2\ll\beta_1$, allowing ϕ_k to be approximated by β_1 in the nonlinearity. With this approximation, which conserves energy, the evolution equations can be written

$$\left[\frac{\partial}{\partial t}+i\omega_j\right]\beta_j=-\sum_{k'}\sum_{m=1}^{2}(-1)^j C_m(k,k')\beta_m'\beta_1'', \tag{14}$$

where the notation $\beta_j'\equiv\beta_j(k',t)$, $\beta_j''\equiv\beta_j(k-k',t)$, $\beta_j\equiv\beta_j(k,t)$ is adopted for shorthand [and also will be applied to the eigenmode frequencies $\omega_j(k)$]. The factors $C_m(k,k')=-(\mathbf{k}'\times\hat{\mathbf{z}}\cdot\mathbf{k})R_m(k')/[R_1(k)-R_2(k)]$ are the *nonsymmetrized* nonlinear coupling coefficients of the eigenmode decomposition. It is straightforward to construct evolution equations for $|\beta_1|^2$, $|\beta_2|^2$, $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$, and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$ by taking appropriate moments of Eq. (14). The result is

$$\left[\frac{\partial}{\partial t}+i\omega_j-i\omega_l^*\right]\langle\beta_j\beta_l^*\rangle=-\sum_{k'}\sum_{m=1}^{2}\left[T_{mjl}(k,k')+T_{mlj}^*(k,k')\right], \tag{15}$$

where $T_{mjl}(k,k')=(-1)^j C_m(k,k')\langle\beta_m'\beta_1''\beta_l^*\rangle$ is a triplet correlation of the eigenmode amplitudes. The indices (j,l) take the values (1,1), (2,2), (1,2), and (2,1) to recover equations for the four correlations.

The moment hierarchy generated by Eqs. (14) and (15), in which moments at a given order are specified by moments of the next order, can be closed using statistical closure theory. Closures based on quasinormal statistics allow $T_{mjl}(k,k')$ to be written as products of second-order moments, closing the hierarchy. The eddy damped quasinormal Markovian closure has been applied to this system,[17] yielding

$$T_{mjl}(k,k') = \frac{(-1)^{j+1}C_m(k,k')}{iW_{m1l}}\sum_{p=1}^{2}\{(-1)^m[C_p(k',k)\langle\beta_p\beta_l^*\rangle \times|\beta_1''|^2 + C_p(k',k'-k)\langle\beta_p''^*\beta_1''\rangle\langle\beta_1\beta_l^*\rangle] - C_p(k-k',-k')\langle\beta_p'^*\beta_m'\rangle\langle\beta_1\beta_l^*\rangle - C_p(k-k',k)\langle\beta_p\beta_l^*\rangle\langle\beta_1'^*\beta_m'\rangle + (-1)^l[C_p^*(k,k') \times\langle\beta_p'^*\beta_m'\rangle|\beta_1''|^2 + C_p^*(k,k-k')\langle\beta_p''^*\beta_1''\rangle \times\langle\beta_1'^*\beta_m'\rangle]\}, \tag{16}$$

where $iW_{m1l}=i\omega_m'+i\omega_1''-i\omega_l^*-\Delta\omega_m'-\Delta\omega_1''-\Delta\omega_l^*$ is the turbulent response function, and $\Delta\omega_m$ is the turbulent (amplitude-dependent) frequency of the eigenmode m. Expressions for $\Delta\omega_1$ and $\Delta\omega_2$ are given in Ref. 17. In computing the flux we will assume that W_{m1l} is dominated by the linear frequencies, consistent with a wave-dominated regime. Wave physics becomes important at long wavelengths because the wave propagation terms carry a lower power of wave number than the nonlinearity. For density evolution the comparison of wave term with nonlinearity yields

$$k_y v_D\hat{\alpha} > k^2 n_k, \tag{17}$$

as the criterion for the wave-dominated regime. The magnitude of the nonlinearity is tied to the linear drive, hence the wave regime tends to be synonymous with weak collisionality $\nu < v_D k_y$. However, the growth rate is also sensitive to proximity to threshold. Near the instability threshold the wave regime is even more strongly enforced. Because the spectra peak at long wavelength the particle flux is dominated by long wavelength modes satisfying Eq. (17). The wave regime is assumed even though the frequency mismatch $\mathrm{Re}[\omega_m'+\omega_1''-\omega_l^*]$ becomes small (of order k^2) when $m=l=1$. Regimes in which $\Delta\omega_m'+\Delta\omega_1''+\Delta\omega_l^*\sim i\omega_m'+i\omega_1''-i\omega_l^*$ are of interest and are considered elsewhere.

Equations (15) and (16) compactly specify the evolution of the spectral densities $|\beta_1|^2$, $|\beta_2|^2$, $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$, and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$ under spectral energy transfer, accounting for the nonlinear mixing of eigenmodes and eigenfrequencies. Wave number convolutions make these equations poorly suited for analytic solution of spectra. However, they can be solved for the dependencies on the physical parameters ν, v_D, η_e, and ϵ, effectively averaging over some range of wave numbers. This remains a difficult problem because the equations are nonlinear, with 48 nonlinear terms for each quantity. While some of these terms are not independent, reducing the number somewhat, it is sufficiently large to make solving Eqs. (15) and (16) in any form nontrivial. To break down the task, these equations were first analyzed for the scaling of the solution with a single normalized parameter $\nu/v_D k_y$, using asymptotic methods based on the smallness of $\nu/v_D k_y$.[17] The task was further simplified by assuming isotropic spectral transfer. Subsequently the high degree of anisotropy in the spectral transfer[25] was taken into account. The anisotropy is so pronounced that the instability saturates by transfer to the narrow layer of zonal modes with $k_y=0$.[23] The anisotropy of the saturation channel means that the problem of wave-number variations cannot be completely separated from the problem of parameter dependencies. Recently, it was shown that this difficulty is amenable to a coarse graining approximation in which the spectrum is split into components for zonal and nonzonal subranges.[23] This led to an asymptotic solution of the leading order scaling in the parameter $\nu/v_D k_y$. The scaling was calculated for eight correlations taken from wave-number averages of $|\beta_1|^2$, $|\beta_2|^2$, $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$, and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$ for $k_y=0$ and $k_y\neq 0$. These were obtained from the eight independent equations extracted from Eq. (15) by taking $k_y=0$ or $k_y\neq 0$, and accounting for the anisotropies in C_1, C_2, and W_{m1l} when $k_y=0$, $k_y'=0$, or $k_y-k_y'=0$.

To construct a meaningful particle flux this process must be taken one step further by resolving the dependence on the driving gradients. The dependence on driving gradients is contained in the parameters v_D and η_e. The eight nonlinear equations for the eigenmode correlations are too complicated to enable a general solution of the dependence on η_e. Instead we expand the equations in the small parameter $\zeta=\hat{\alpha}-1=\frac{3}{2}\eta_e$. Taking ζ small reduces the number of nonlinear terms in the eight equations, and simplifies the solution. Near-threshold values of η_e are consistent with the idea that strong transport above the threshold reduces the temperature gradient and thereby keeps it near threshold. Since there are three small parameters, $\delta=\nu/v_D k_y$, k^2, and ζ, we must adopt an ordering scheme for consistency. The ordering we assume is

$$\zeta \sim \sqrt{\delta} \sim \sqrt{k}. \tag{18}$$

In the TEM model, Eqs. (3) and (4), the collisionless trapped electron mode is unstable for $\zeta > (1+k^2)^{-1}-1 \approx -k^2$. Hence, for this ordering the system is above threshold with $\eta_e > 0$, but not significantly so, because η_e does not exceed unity. This ordering is by no means the only possibility for $\zeta \ll 1$ within a long wavelength, weakly collisional regime. Smaller and larger ζ values are of interest. For $\zeta\sim 1$ we have not been able to find analytic solutions; for smaller ζ the system is closer to threshold, with smaller transport and fluctuation levels. Increasing k to $k\sim\sqrt{\delta}$ was found to have no significant change on saturation scalings.

The solution of the problem consists in finding the lowest-order asymptotic dependence of the eight spectra on the parameter ζ. This is done by examining asymptotic balances and eliminating those that are not consistent, with the criteria for consistency given below. Starting from the prior expansion in $\delta=\nu/k_y v_D$, the spectra have the form

$$|\beta_1(k)|^2|_{k_y\neq 0} = k_y^2 v_D^2 S_{11T}\zeta^{\alpha_1}, \tag{19}$$

$$|\beta_2(k)|^2|_{k_y\neq 0} = \nu^2 S_{22T}\zeta^{\alpha_2}, \tag{20}$$

$$\mathrm{Re}\langle\beta_1^*(k)\beta_2(k)\rangle|_{k_y\neq 0} = \nu^2\,\mathrm{Re}\,S_{12T}\zeta^{\alpha_3}, \tag{21}$$

$$\mathrm{Im}\langle\beta_1^*(k)\beta_2(k)\rangle|_{k_y\neq 0} = \nu k_y v_D \,\mathrm{Im}\, S_{12T}\zeta^{\alpha_4}, \tag{22}$$

$$|\beta_1(k)|^2|_{k_y=0} = \bar{k}_y^2 v_D^2 S_{11Z}\zeta^{\alpha_5}, \tag{23}$$

$$|\beta_2(k)|^2|_{k_y=0} = \bar{k}_y^2 v_D^2 S_{22Z}\zeta^{\alpha_6}, \tag{24}$$

$$\mathrm{Re}\langle\beta_1^*(k)\beta_2(k)\rangle|_{k_y=0} = \bar{k}_y^2 v_D^2 \,\mathrm{Re}\, S_{12Z}\zeta^{\alpha_7}, \tag{25}$$

$$\mathrm{Im}\langle\beta_1^*(k)\beta_2(k)\rangle|_{k_y=0} = \nu\bar{k}_y v_D \,\mathrm{Im}\, S_{12Z}\zeta^{\alpha_8}, \tag{26}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_8$ are the as yet unknown scaling exponents of ζ to be determined by the consistency of asymptotic balances.

These spectra are the solutions of eight independent equations constructed from Eq. (15). Evolution equations for $|\beta_1|^2|_{k_y\neq 0}$ and $|\beta_1|^2|_{k_y=0}$ are obtained by taking $(j,l)=(1,1)$ with $k_y=0$ and $k_y\neq 0$. The condition $k_y\neq 0$ requires special attention for anisotropic transfer involving modes with $k_y'=0$. Modes with $k_y-k_y'=0$ replicate terms already generated by $k_y'=0$ and need not be treated separately to uncover the scaling exponents α_j. Evolution equations for $|\beta_2|^2|_{k_y\neq 0}$ and $|\beta_2|^2|_{k_y=0}$ are obtained by taking $(j,l)=(2,2)$, again with $k_y=0$ and $k_y\neq 0$. Evolution equations for $\mathrm{Re}\langle\beta_1^*\beta_2\rangle|_{k_y\neq 0}$ and $\mathrm{Re}\langle\beta_1^*\beta_2\rangle|_{k_y=0}$ are obtained by adding together the equations for $(j,l)=(1,2)$ and $(j,l)=(2,1)$, evaluated at $k_y=0$ and $k_y\neq 0$. The equations for $\mathrm{Im}\langle\beta_1^*\beta_2\rangle|_{k_y\neq 0}$ and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle|_{k_y=0}$ are obtained from the difference of the equations for $(j,l)=(1,2)$ and $(j,l)=(2,1)$, again evaluated at $k_y=0$ and $k_y\neq 0$. To generate these equations C_1, C_2, and W_{ml1} are expanded asymptotically for $\zeta\sim\sqrt{\delta}\sim\sqrt{k}\ll 1$. The coefficients C_1 and C_2 appear with wave-number dependencies $C_1(k,k')$, $C_1(k',k)$, $C_1(k'',k)$, $C_2(k,k')$, $C_2(k,k'')$, $C_2(k',k)$, $C_2(k'',k)$, $C_2(k',-k'')$, and $C_2(k'',-k')$, and must be evaluated separately for the three conditions $k_y=0$, $k_y'\neq 0$; $k_y\neq 0$, $k_y'=0$; and $k_y\neq 0$, $k_y'\neq 0$. Likewise W_{111}, W_{112}, W_{211}, and W_{212} must be expanded asymptotically in δ and ζ for $k_y=0$, $k_y'\neq 0$; $k_y\neq 0$, $k_y'=0$; and $k_y\neq 0$, $k_y'\neq 0$.

The evolution equations for $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ can be solved exactly, even before taking the asymptotic expansion, by exploiting symmetries of $T_{mjl}(k,k')$. In particular, $T_{m12}=-T_{m22}$ and $T_{m21}=-T_{m11}$, which creates a simple relationship between the evolution equations for $|\beta_1|^2$ and $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$. In steady state this relationship leads to the identity

$$-\mathrm{Re}(\omega_1^*-\omega_2)\mathrm{Im}\langle\beta_1^*\beta_2\rangle - \mathrm{Im}(\omega_1^*-\omega_2)\mathrm{Re}\langle\beta_1^*\beta_2\rangle + \mathrm{Im}\,\omega_2|\beta_2|^2 + \mathrm{Im}\,\omega_1|\beta_1|^2 = 0. \tag{27}$$

Imposing the asymptotic ordering and taking $k_y=0$, this identity reduces to

$$\mathrm{Re}\, S_{12Z} = -S_{22Z}, \qquad \alpha_7=\alpha_6, \tag{28}$$

while for $k_y\neq 0$ it reduces to

$$\mathrm{Im}\, S_{12T} = \frac{\epsilon^{1/2}}{(1-\epsilon^{1/2}\hat{\alpha})^2} S_{11T}, \qquad \alpha_4 = 1+\alpha_1. \tag{29}$$

The remaining six exponents were evaluated by testing asymptotic balances for all possible combinations of α_j ranging from 0 to 2. There are six independent exponents with three possible values, hence there are $3^6=729$ possible ζ scalings to test. Consistent asymptotic balances are defined as those satisfying the following physical constraints:

1. In the equation for $|\beta_1|^2|_{k_y\neq 0}$, the linear term must enter the lowest-order balance, along with at least one nonlinear term. This constraint ensures that the turbulence is driven by the TEM instability, and that there is a saturated steady state in which the instability drive is balanced by turbulent energy transfer.
2. In the equation for $|\beta_2|^2|_{k_y=0}$, the linear term must enter the lowest-order balance, along with at least one nonlinear term. Otherwise, S_{22Z} decays if the linear term is the only lowest-order term, or, the system saturates at an unrealistically high level. Previous work has shown that the dominant saturation channel is energy transfer to zonal modes on the damped eigenmode branch.[23] Other channels, e.g., energy transfer to viscously damped high k modes on the unstable eigenmode branch, are much less efficient. They are important only if the transfer to zonal modes is artificially suppressed, leading to much higher fluctuation levels.
3. In the equation for $|\beta_2|^2|_{k_y\neq 0}$, the linear term must not be the only term in the lowest-order balance. Otherwise, S_{22T} decays. The linear term can be of higher order than nonlinear terms, provided there are at least two independent nonlinear terms of opposite sign.
4. In the equation for $\mathrm{Im}\langle\beta_1^*\beta_2\rangle|_{k_y=0}$, the linear term must not be the sole lowest-order term. Otherwise, $\mathrm{Im}\, S_{12Z}$ decays.
5. In the equation for $\mathrm{Im}\langle\beta_1^*\beta_2\rangle|_{k_y\neq 0}$, the linear term must not be the sole lowest-order term. Otherwise, $\mathrm{Im}\, S_{12T}$ decays.

The sixth condition, applying to the equation for $|\beta_1|^2|_{k_y=0}$, is discussed below. In these conditions the leading order nonlinear terms must not be linearly dependent and cancel one another when $k_y=0$ or $k_y'=0$. If any of the constraints 1–5 listed above fail, the set of numerical values being tested for the scaling exponents $\alpha_1,\alpha_2,\ldots,\alpha_8$ is deemed a failure and cannot represent a solution. These constraints are physically reasonable, provided the damped eigenmode is excited to finite amplitude and does not simply decay as usually assumed. These six conditions can be inferred from numerical solutions and taken as empirical; however, they can also be deduced from analysis.[18]

Of the 729 scalings satisfying Eqs. (27) and (29), only one satisfies all five conditions above. This scaling is $\alpha_1=1$, $\alpha_2=2$, $\alpha_3=1$, $\alpha_4=2$, $\alpha_5=1$, $\alpha_6=2$, $\alpha_7=2$, and $\alpha_8=0$. In the equation for $|\beta_1|^2|_{k_y=0}$, this scaling eliminates all but one nonlinear term from the lowest order. This term drives the nonlinear excitation of the zonal flow $|\beta_1|^2|_{k_y=0}$. However, as Eq. (15) stands, there is no linear term to balance this zonal flow drive because $\omega_1|_{k_y=0}\equiv 0$. Even though zonal modes on the damped branch saturate the unstable $k_y\neq 0$ modes, the zonal flow $|\beta_1|^2|_{k_y=0}$ does not reach a steady state when the system is close to the instability threshold with $\eta_e\sim\sqrt{\nu/v_D k_y}$. (Previous work showed that when $\eta_e\sim 1$ nonlinear transfer into and out of the zonal flow by independent nonlinear terms was in balance, yielding a steady state that required no damping of the zonal flow.[23]) However, when the system is close to threshold, steady state requires additional damping on $|\beta_1|^2|_{k_y=0}$ not included in Eqs. (3) and (4). This damping goes as ζ^3 and is smaller than the damping rate of the damped branch.

The assumption of proximity to threshold, $\zeta\ll 1$, raises the order of the lowest-order balance and considerably simplifies Eqs. (15) and (16). An algebraic solution is now obtainable, given by

$$S_{11T}\zeta^{\alpha_1}=\frac{4\epsilon^{1/2}\delta^2\zeta}{(\bar{k}_y\bar{k}_x)^2\bar{k}^2}, \tag{30}$$

$$\mathrm{Im}\,S_{12T}\zeta^{\alpha_4}=\frac{4\epsilon\delta^2\zeta^2}{(\bar{k}_y\bar{k}_x)^2\bar{k}^2(1-\epsilon^{1/2})}, \tag{31}$$

$$\mathrm{Re}\,S_{12T}\zeta^{\alpha_3}=\frac{4\epsilon(1-\epsilon^{1/2}+2\epsilon-\epsilon^{3/2})\delta^2\zeta}{(\bar{k}_y\bar{k}_x)^2\bar{k}^2(1-\epsilon^{1/2})^2\left[(1-\epsilon^{1/2})^2-\dfrac{1}{2C_z}\right]}, \tag{32}$$

$$S_{22T}\zeta^{\alpha_2}=\frac{8\epsilon(1-\epsilon^{1/2}+2\epsilon-\epsilon^{3/2})\delta^2\zeta^2}{(\bar{k}_y\bar{k}_x)^2\bar{k}^2(1-\epsilon^{1/2})^2\left[(1-\epsilon^{1/2})^2-\dfrac{1}{2C_z}\right]}, \tag{33}$$

$$S_{11Z}\zeta^{\alpha_5}=\frac{4\epsilon^{1/2}(1-\epsilon^{1/2}+2\epsilon-\epsilon^{3/2})\delta^2\zeta}{C_z(\bar{k}_y\bar{k}_x)^2\bar{k}^2\left[(1-\epsilon^{1/2})^2-\dfrac{1}{2C_z}\right]}, \tag{34}$$

$$\mathrm{Re}\,S_{12Z}\zeta^{\alpha_7}=-\frac{4\epsilon(1-\epsilon^{1/2}+2\epsilon-\epsilon^{3/2})\delta^2\zeta^2}{(\bar{k}_y\bar{k}_x)^2\bar{k}^2(1-\epsilon^{1/2})^2\left[(1-\epsilon^{1/2})^2-\dfrac{1}{2C_z}\right]}, \tag{35}$$

$$\mathrm{Im}\,S_{12Z}\zeta^{\alpha_8}=-\frac{16(1-\epsilon^{1/2}+2\epsilon-\epsilon^{3/2})}{(\bar{k}_y\bar{k}_x)^2(1-\epsilon^{1/2})\left[(1-\epsilon^{1/2})^2-\dfrac{1}{2C_z}\right]}, \tag{36}$$

$$S_{22Z}\zeta^{\alpha_6}=\frac{4\epsilon(1-\epsilon^{1/2}+2\epsilon-\epsilon^{3/2})\delta^2\zeta^2}{(\bar{k}_y\bar{k}_x)^2\bar{k}^2(1-\epsilon^{1/2})^3\left[(1-\epsilon^{1/2})^2-\dfrac{1}{2C_z}\right]}, \tag{37}$$

where $C_z>\frac{1}{2}(1-\epsilon^{1/2})^{-2}$ is the order-unity coefficient of the additional damping of $|\beta_1|^2|_{k_y=0}$, $\bar{k}^2$ is a spectrum average of the quantity $[k^2-(k-k')^2]$ (with averages over both k and k'), and $\bar{k}_x$ and $\bar{k}_y$ are typical wave numbers in the x and y directions. The average that yields $\bar{k}_y$ does not include $k_y=0$. Moreover, by virtue of the saturation constraints enumerated above, the wave-number average is restricted to the subrange over which the growth remains strong. The additional damping is given by $\gamma=C_z\zeta\nu$. The scalings evident in Eqs. (30)–(37), and in particular the gradient dependencies, are different from those previously given.[23] Previous work did not resolve the dependence on η_e. In the expressions for $|\beta_1|^2$, $|\beta_2|^2$, $\mathrm{Re}\langle\beta_1\beta_2^*\rangle$, and $\mathrm{Im}\langle\beta_1\beta_2^*\rangle$ there were no factors ζ^{α_j} because the functions S_{11T}, S_{22T}, $\mathrm{Re}\,S_{11T}$, $\mathrm{Im}\,S_{12T}$, S_{11Z}, S_{22Z}, $\mathrm{Re}\,S_{11Z}$, and $\mathrm{Im}\,S_{12Z}$ were order-unity functions with unspecified dependence on η_e and ϵ. Note that the order-unity factor $\delta^2/\bar{k}^2$ that appears in all but one of Eqs. (30)–(37) significantly modifies the initial scaling factors of $k_y^2 v_D^2$, $\bar{k}_y^2 v_D^2$, ν^2, $\nu k_y v_D$, and $\nu\bar{k}_y v_D$ in Eqs. (19)–(26). This is because $\zeta\ll 1$ raises the order of many terms and changes the dominant spectrum balances. Consequently, it must be remembered that the gradient dependencies of Eqs. (30)–(32) and (34)–(37), and therefore the gradient dependencies of inward and outward components of the particle flux, apply to the near-threshold regime assumed, and, in particular, for the ordering $\zeta\sim\sqrt{\delta}\sim\sqrt{k}\ll 1$. A different ordering will likely yield different dependencies.

IV. PARTICLE FLUX

The particle flux is evaluated for the near-threshold saturated turbulent state given by Eqs. (30)–(37) by substituting into Eq. (9). Only the spectra for nonzonal wave numbers contribute to the flux. However, because the spectral transfer to zonal wave numbers is so large, the zonal wave numbers are central to the nonzonal spectra entering the sum. Carrying out the indicated operations,

$$\Gamma\sim\frac{4\epsilon^{1/2}\nu^3\zeta}{(1-\epsilon^{1/2})v_D(\bar{k}_x\bar{k}_y)^2\bar{k}^2}\left\{\frac{(2-\epsilon^{1/2})\zeta}{(1-\epsilon^{1/2})}-\frac{(1+2\zeta)(1-\epsilon^{1/2}+2\epsilon-\epsilon^{3/2})}{\left[(1-\epsilon^{1/2})^2-\dfrac{1}{2C_Z}\right]}\right\}. \tag{38}$$

The first term is outward. It includes the quasilinear flux from $|\beta_1|^2$ and the nonquasilinear contribution from $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$. The second term is inward and is comprised of the nonquasilinear contributions from $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ and $|\beta_2|^2$. Several features of this expression are noteworthy. Foremost, inward flux contributions, which arise exclusively from nonquasilinear effects, are significant. The contribution from $|\beta_2|^2$, which is a negative-definite component inexorably tied to nonlinear damped eigenmode excitation, is similar in magnitude to the quasilinear flux. The inward component from $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ is even larger. While the spectrum $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ is smaller than its quasilinear counterpart $|\beta_1|^2$ by a factor $\delta^2\epsilon^{1/2}$, the coefficients of these spectra in the flux expression have a ratio $\mathrm{Im}\, R_2/\mathrm{Im}\, R_1$, which goes as $(\delta^2\epsilon^{1/2}\zeta)^{-1}$. Consequently the $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ term is larger than the quasilinear flux. This term does not have a definite sign, and as argued below, is probably diminished by presently unresolved wave-number contributions near $k_y=0$ of opposite sign. The same effect applies to the outward component from $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$, and means that the negative sign of the three nonquasilinear components in Eq. (38) can be expected to persist even when saturation is fully resolved in wave number. The net negative sign of the three nonlinear terms was observed in simulations,[17] although the system was well above threshold and therefore in a different regime.

A second important feature of Eq. (38) is the presence of two distinct gradient scalings. The quasilinear term and the terms proportional to $|\beta_2|$ and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$ scale as L_n^3/L_T^2, whereas the larger term proportional to $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ scales as L_n^2/L_T. Thus, the role of the damped eigenmode is not simply the introduction of a component that reduces the quasilinear flux but otherwise leaves its scaling unchanged. On the contrary, the nonlinear flux components must be viewed as independent because of distinct scalings. The third feature of interest is that no scaling of Eq. (38) is compatible with a flux that can be modeled as the sum of diffusive and convective components. The saturation of the instability is such that every term of the flux, including the quasilinear term, is highly nondiffusive. The transport is stronger for flat density gradients. It should be remembered that the gradient scaling, while interesting, is sensitive to the ordering assumption that defines the threshold regime studied here. In different regimes the scaling is expected to change. For example, when k^2 becomes order unity, the threshold factor in the $|\beta_1|^2$ term of Eq. (9) now has leading order components that do not depend on η_e.

For the present ordering with $\zeta<1$, the inward component from $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ has the largest magnitude because it goes as ζ while the remaining terms go as ζ^2. This flux approximation therefore can assume negative values, violating the positivity constraint of Eq. (12). As noted, the constraint is global, whereas the flux computed in Eq. (38) is taken from spectrum balances that apply locally to restricted wave-number subranges. If the balances could be solved for arbitrary wave number (a daunting task given the convolutions), the resulting spectral densities for $|\beta_1|^2$, $|\beta_2|^2$, $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$, and $\mathrm{Im}\langle\beta_1^*\beta_2\rangle$ would yield a flux that satisfies the positivity constraint. However, the solutions given in Eqs. (30)–(32) and (34)–(37) are restricted to a limited subrange of wave numbers, over which the instability drive remains fairly constant. The failure of Eq. (38) to reflect the positivity constraint should not be taken as indication that the conclusions of the previous paragraphs regarding the inward flux components are not valid. The inward components are a robust and significant feature of simulations, which nevertheless also show that $\Gamma>0$ when $dW/dt\geqslant 0$. One possible source for the discrepancy between Eqs. (38) and (12) is the wave-number variation of $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ inferred from Eqs. (32) and (35). According to these expressions $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ changes sign as $k_y\rightarrow 0$. This behavior is also robust: the negative sign of $\mathrm{Re}\langle\beta_1^*\beta_2\rangle|_{k_y=0}$ follows directly from the identity of Eq. (27). A positive value away from $k_y=0$ was found in simulations, but is also obtained from the spectrum balances, both in the threshold calculation described here, and for η_e of order unity.[17] We expect the spectra to vary smoothly so that $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ first decreases as k_y approaches zero, and then passes through zero for k_y finite before reaching the negative value given by Eq. (35). This would make $\mathrm{Re}\langle\beta_1^*\beta_2\rangle$ smaller in a spectrum sum than its value taken from a pointwise balance near its maximum. The latter typifies Eq. (38), while a spectrum sum picking up negative values slightly above $k_y=0$ enters the exact flux.

V. CONCLUSIONS

This paper demonstrates that there is an inward particle flux component associated with the nonlinear excitation of a damped eigenmode to finite-amplitude levels. The resulting nonlinear mixing of unstable and damped eigenmodes cannot be predicted within the quasilinear approximation. Hence the inward flux component arising from the damped eigenmode is a nonlinear effect beyond quasilinear theory. The damped eigenmode affects the particle transport by changing the cross phase between the density and the potential from the value stipulated according to the quasilinear prescription by the unstable eigenmode. The excitation of a damped eigenmode answers the question: when does the quasilinear approximation for the transport correlation break down? It does so in a way that is more descriptive than merely saying that the eigenmode becomes nonlinear or that the cross phase is no longer given by the quasilinear value. Moreover tracking the nonlinear evolution of all the eigenmodes in a complete basis constitutes a systematic method for calculating the full nonlinear particle flux.

A simple fluid model for trapped electron mode turbulence provides a concrete application of this approach, allowing analytic calculation of inward particle flux components beyond quasilinear theory. The calculation requires the solution of spectrum evolution equations for eigenmode autocorrelations and cross correlations. This is undertaken using a joint asymptotic expansion in ratio of collision frequency to diamagnetic frequency, wave number, and η_e^2, where η_e is the ratio of density to temperature gradient scale lengths. In this regime the system is near the instability threshold, i.e., it is weakly driven. There are inward flux components of significant magnitude arising from both the autocorrelations and cross correlations of the eigenmodes. All flux components are nondiffusive. The quasilinear flux component, and components proportional to the autocorrelation of the damped eigenmode and imaginary part of the cross correlation scale as L_n^3/L_T^2. The real part to the cross correlation scales as L_n^2/L_T. These scalings are not universal but depend on the ordering of the small parameters in the asymptotic expansion. These parameters delineate wavelength, collisionality, and instability threshold regimes.

The primary significance of the present calculation is its identification and description of new physics affecting particle transport, and in particular, inward transport. The details of the calculation, e.g., the gradient scalings, are of secondary significance. It is important to show that they can be derived, and that scalings associated with damped eigenmode excitation can differ from those of the quasilinear flux. However, the details are sensitive to parameter regimes. Only one of several possible regimes has been examined and we have not attempted to match parameters to experimental conditions. Indeed, the model is simple, designed to capture key linear and nonlinear effects and allow careful examination of complex nonlinear physics, to the exclusion of other details. As mentioned, we have not considered weakly collisional trapped electron dynamics in conjunction with ion instability such as ITG. It is well known that electron dynamics which of themselves produce only outward transport in unstable situations, drive inward transport when they are a stabilizing component of an instability supported primarily by ion dynamics.[2,3] This means that the type of physics described herein should be examined as a nonadiabatic component of ITG turbulence, to determine if the inward mixing-mode flux becomes larger than the quasilinear value. It should also be observed that while the present calculation involves a thermal force, the effect of damped eigenmode excitation is more general, and where relevant, can be expected to modify curvature-driven pinches as well.

The prevalence of damped eigenmode excitation, upon which these inward transport effects are predicated, is an open question. Damped eigenmode excitation has been observed in the present model, in Rayleigh-Taylor turbulence,[19] and in a model for ion drift wave turbulence,[18] with significant effects on transport evident in all three cases. On the other hand, a numerical study of trapped electron turbulence using a gyrokinetic model shows little deviation from the linear state.[26] While general criteria have been formulated for the excitation of stable eigenmodes and for their role as significant participants in transport,[18] such criteria must be checked on a case by case basis. Moreover, they are rooted in eigenmode space, for which, as yet, there is limited experience and intuition. This makes it difficult to predict *a priori* which types of turbulence might be more susceptible to damped eigenmode excitation. We note that geodesic acoustic modes and zonal flows appear as fluctuations that are distinct from the unstable eigenmode in models where nonadiabatic electrons are not intrinsic to the instability. The present calculation demonstrates that it is a misconception to view these types of fluctuations as examples of damped eigenmode excitation. Geodesic acoustic modes and zonal flows make no direct contribution to the flux because they have $k_y=0$, whereas the damped eigenmode described herein produces large inward flux components because it has a full spectrum of fluctuations with $k_y \neq 0$. Nonetheless, zonal modes are the primary attractor of spectrally transferred energy. The nonzonal, damped eigenmodes provide part of the conveyance and, to some extent, simply share in the wealth of the transfer anisotropy. All of this is accounted for in the solution of the spectrum evolution equations.

The trapped electron model describes damped-eigenmode effects in which the cross phase between the unstable and stable eigenmodes tends to a stationary value.[17] This makes the particle flux behavior simpler than what it is for cases in which cross phases lock only transiently.[18] There the flux is intrinsically nonstationary and intermittent, even if the gradients are held fixed. As with the present case, the origin of the effect lies in the excitation of stable eigenmodes. It is important that future study of particle transport look not just at the nonquasilinear and nondiffusive effects studied herein, but at nonstationary effects, both those tied to evolving profiles, and to damped eigenmode excitation.

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