Two-fluid tearing instability in force-free magnetic configuration

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In general, the linear two-fluid tearing instabilities are driven by shear Alfvén (SA), compressional Alfvén (CA), and slow magnetoacoustic (MA) modes modified on short scales by two-fluid effects. Previous two-fluid theories were devoted to either the hot plasma case where coupling of the SA and the MA waves dominates, or to the cold plasma limit, $\beta=0$, where the instability is driven by the SA and the CA waves. Taking into account plasma compressibility and the Hall term, we derive general tearing equations that cover the two limiting cases and the transition between them. In particular, in the hot plasma case, equations are derived that depend on the factor $\beta/(1+\beta)$ and span the validity of resistive to electron MHD. The important effect of resistive diffusion of the "out-of-plane" component of the magnetic field perturbation $B_{\parallel}^{(1)}$ is also included. Two new solutions where this effect dominates are obtained within the scope of the hot plasma model. Whistler scaling $\gamma \propto \Delta'^2$ is found for the collisionless tearing mode instead of the kinetic Alfvén scaling $\gamma \propto \Delta'$ at small Δ'. Previous calculations for coupling with the MA waves at large Δ' did not take the diffusion of $B_{\parallel}^{(1)}$ into account. The joint effect of these factors is presented in the work. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1773778]

I. INTRODUCTION

Tearing instabilities play an important role in fusion experiments and astrophysical applications. They are thought to be responsible for a variety of physical phenomena including fast reconnection of magnetic fields, relaxation to the Taylor state, the dynamo effect, the formation of magnetic islands, and anomalous radial transport in stochastic magnetic fields. At higher plasma temperatures, the viability of standard resistive MHD models[1,2] for the plasma dynamics becomes questionable. This is especially important for tearing modes, where the spatial structure of eigenfunctions near the resonant surface (in the linear tearing layer) is determined by electron skin depth which is normally much shorter than ion-sound gyroradius ρ_s or ion skin depth c/ω_{pi}. The smallness of the electron skin depth in comparison with the ion scales leads to a decoupling of electrons from ions in the vicinity of the reconnection layer, speeding up of the instability and a broadening of the tearing layer. The enhanced growth rate is caused by fast, vortex-like motion of the decoupled electrons in the plane perpendicular to the guiding magnetic field (reconnection plane). This provides enhanced transport of the magnetic flux toward the diffusive layer. This transport is much faster than in the single-fluid MHD case where electrons are coupled to ions due to their joint $\mathbf{E}\times\mathbf{B}$ drift. Only small perpendicular currents are generated by polarization drifts.

Large-amplitude perpendicular eddy currents appear in two-fluid regimes due to electron–ion decoupling on short scales. They are driven by magnetic field perturbations parallel to the guiding magnetic field (out-of-plane component) that are small in single-fluid MHD, but are of major importance in two-fluid theory. In this work, we derive the basic equation that connects $B_{\parallel}^{(1)}$ with the perpendicular (radial) component of the perturbed magnetic field. It describes previously investigated regimes of electron–ion decoupling as well as the new ones in which two-fluid effects are important. Two-fluid physics enters the equations via the ion skin depth $d_i=c/\omega_{pi}$ and ion-sound gyroradius $\rho_s=c_s/\omega_{Bi}$, where c_s is the ion-sound speed and ω_{Bi} is the ion cyclotron frequency calculated with the guiding magnetic field that is present in high-temperature plasma experiments but absent in many theoretical treatments of Hall-MHD and collisionless reconnection.

In the context of two-fluid theory, the Hall term by itself cannot provide reconnection because the magnetic field is frozen into the electron fluid. The collisionless effect of electron inertia $d_e=c/\omega_{pe}$ and finite resistivity η are taken into account as a means to break the frozen flux theorem. Their combined effects are characterized by the sum of the collisionless and resistive electron skin depths $\delta^2=d_e^2+c^2\eta/4\pi\gamma$ where γ is the growth rate of tearing instability. Thus, finite-electron-inertia modification of the parallel Spitzer resistivity is taken into account but no other kinetic physics related to the Dopler shift $k_{\parallel}v_{Te}$ and associated electron Landau damping features are included. In accordance with the definition of δ, this factor is γ dependent and, therefore, initially unknown. Without specification of its exact value just by using the smallness of δ in comparison with the ion scales ρ_s and d_i, one can obtain the dispersion relation in terms of $\delta(\gamma)$ and then solve it for γ explicitly. This allows us to cover both collisionless and semicollisional regimes with one universal approach.

An extreme limiting case of the ion–electron decoupling is described by electron magnetohydrodynamics (EMHD), where ion motion is ignored. In many cases of practical interest, intermediate regimes are realized where both electrons and ions contribute to the dynamics of instability. In a contrast to complicated kinetic treatments (see, for example, Refs. 3–6), relatively simple and effective fluid-based approaches were developed for tearing instabilities in early 1990s. The corresponding fluid-based treatments can be subdivided into two groups. The first group (see, for example, Refs. 7–11) is devoted to the cold plasma model, $\beta=0$. In this limit, all temperature-dependent terms are dropped, and the dynamics of the instability is determined by δ and d_i. At large $d_i \gg L$, where L is the scale of the equilibrium magnetic field, ion motion can be ignored, and the instability exhibits the properties of the pure electron whistler mode. We will refer to this case as a whistler-mediated tearing instability.

The second group is devoted to an opposite limit of hot plasma, where thermal pressure and particle gyromotion determine the dynamics of the instability. This case is referred to later to as a kinetic Alfvén-driven tearing instability. Generally speaking, the ion response to electric fields with transverse scales shorter than the ion-Larmor radius should be described by the kinetic theory. However, single-fluid MHD calculations with the Hall term[16] as well as two-fluid equations of the Braginskii type with plasma compressibility[15] show that the joint effect of the Hall term and plasma compressibility yields the results similar to kinetic ion models.[17,18] Some qualitative arguments in favor of this agreement are given in Sec. II. Various linear regimes of the "hot" tearing instability were analyzed in Ref. 12 (hot electrons and cold ions), in Ref. 13 within the scope of the four-field model,[14] and in Ref. 15 with the use of two-fluid MHD theory. They yield the growth rate and the structure of the eigenfunctions in terms of δ, ρ_s, d_i, and stability factor Δ'. The stability factor Δ' is a measure of free energy available for resistive reconnection.[1] It is calculated from the solution of the marginal ideal MHD equations in the outer region. The quantity Δ' is an asymptotic matching parameter defined as the jump in the logarithmic derivative of the radial magnetic field across the reconnection layer. We will consider Δ' as given and analyze the structure of the inner layer in the limits of small and large Δ'. The two-fluid effects and ion–electron decoupling are important at $\rho_s, d_i \gg \delta$, otherwise the single-fluid MHD theory is valid. These basic inequalities are taken to be satisfied in the following calculations.

Two different models of cold and hot plasma and the gap between them raise a question about their interrelation and conditions of applicability. This motivated our interest in the construction of a more general theory that covers both cases and describes the transition between them. We analyze slab geometry for a force-free plasma equilibrium with uniform density, temperature, and pressure profiles for electrons and ions, assuming that the equilibrium magnetic field consists of small shearing component $B_y^{(0)}(x)$ and large guiding field $B_z^{(0)}$. The perturbed quantities are taken to be functions of x and y. Assuming no equilibrium pressure gradients, we treat the absolute value of the pressure as an arbitrary parameter and, correspondingly, vary β in a range $0 \leqslant \beta < \infty$. Such an equilibrium may exist locally (within a certain radial extension) or can be supported by distant material walls. This allows us to exclude the effects of diamagnetic flows assuming that $\gamma \gg \omega_{*e,i}$, where $\omega_{*e,i}$ are electron and ion diamagnetic frequencies. Thus, we study two-fluid effects caused by the ion–electron decoupling, while the two-fluid effects driven by the equilibrium diamagnetic flows are ignored. They are considered, for example, in Refs. 19 and 20.

The perturbations of ion and electron pressures are essentially nonuniform. They are driven by plasma compressibility, $\nabla \cdot \mathbf{v} \neq 0$, and are treated to be either adiabatic or isothermal. Plasma dynamics is described by the momentum equation (the sum of ion and electron equations of motion) and generalized Ohm's law (electron momentum equation) without viscous and gyroviscous effects. Using the small parameter $\epsilon = B_y^{(0)}/B_z^{(0)} \ll 1$, a vorticity equation is derived as well as an induction equation for a parallel component of the vector potential. The induction equation has the same structure as in resistive MHD theory with the replacement $\mathbf{v} \rightarrow \mathbf{v}_e$, reflecting the fact that the magnetic field is frozen in the electron fluid. Making the decomposition $\mathbf{v}_e = \mathbf{v} - (1/ne)\mathbf{j}$ yields the Hall term in the induction equation. An Ampère's law is used to express $\mathbf{j}_\perp$ as a function of the out-of-plane perturbations $B_z \simeq B_\parallel^{(1)}$. In order to close the system of equations, we derive the relationship between B_z and the perpendicular component B_x. This equation is the main result of the paper. It is derived in Sec. II by expressing the plasma compressibility $\nabla \cdot \mathbf{v}$ from the perpendicular component of the induction equation and substituting into the equation for $\nabla \cdot \mathbf{v}$ which follows from the plasma momentum equation.

In Sec. III, the tearing equations are applied for the case of waves propagating in a uniform magnetic field. The analysis shows that the two-fluid tearing instability is driven by shear Alfvén (SA), compressional Alfvén (CA), and slow magnetoacoustic (MA) waves modified on short scales by two-fluid effects. This classification of the modified waves is based on their behavior in the long wavelength MHD limit.

In Sec. IV A, the general set of tearing equations is simplified in the cold plasma limit, when the time of propagation of the ion-sound wave across the reconnection layer is much longer than the time scale of the instability. The reduced equations coincide with the equations derived in Ref. 11, and corresponds to the interaction of the SA and the CA modes while the MA mode is decoupled. On short scales, $kd_i \gg 1$, the CA branch converts into the electron whistler mode.

In the opposite case of hot plasma, the ion-sound wave propagation time is much shorter than the time scale of the instability. It provides equilibration of total (magnetic + thermal) pressures across the layer and, correspondingly, yields $B_z/B_z^{(0)} \propto \beta$. This situation is discussed in Sec. IV B, and is of the main interest for magnetically confined plasmas. The two-fluid tearing instability is driven by the SA and the MA modes while the CA mode is decoupled. Ion–electron decoupling on short scales leads to the mode dispersion typical for kinetic Alfvén waves, $\omega \propto k_\parallel k \rho_s^2$. A formal expansion of the general equation for B_z at $\gamma \ll c_s/L$ yields tearing equations similar to those analyzed in Ref. 13. They are derived in a more general form with the dependence on β in the form $\beta/(1+\beta)$ that is universally applicable for wide range $0 < \beta < \infty$. This important property of the hot plasma case was first treated in Ref. 21 in terms of the universal scale $R = \rho_s d_i / \sqrt{\rho_s^2 + d_i^2}$.

Our equations also contain the important effect of diffusion of the perturbed B_z. This effect was not properly accounted for in Ref. 13 where the process of coupling of the SA and the MA waves was investigated at finite β when it effectively reduces the out-of-plane component B_z and, thus, suppresses two-fluid effects. This mechanism is reconsidered in Sec. V, which is devoted to the solutions of the tearing equations in the hot plasma case. We show that diffusion of B_z leads to an additional suppression of the two-fluid effects. At large values of the stability factor Δ', the contribution from this mechanism is equally important with the contribution from the MA wave and yields an additional reduction of the growth rate by the same factor as calculated in Ref. 13. At small Δ', it is shown that due to the narrow width of the tearing layer, the enhanced diffusion of B_z leads to the conversion of the kinetic Alfvén mode into the whistler-mediated regime of tearing instability.

II. THE GEOMETRY OF THE PROBLEM AND BASIC EQUATIONS

The two-fluid approach is based on the fluid momentum balance equation

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla\left(p+\frac{B^2}{8\pi}\right)+\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}, \tag{1}$$

and the generalized Ohm's law

$$\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B} = \frac{1}{nec}\mathbf{j}\times\mathbf{B}-\frac{\nabla p_e}{ne}-\frac{m_e}{e}\frac{d\mathbf{v}_e}{dt}+\eta\mathbf{j}, \tag{2}$$

which describes the balance of the electron momentum. In this work, viscous effects are ignored, $\nabla\cdot\pi_s=0$, for both electrons and ions. Ion velocity $\mathbf{v}_i$ is associated with the fluid velocity $\mathbf{v}$ (center of mass velocity), $\mathbf{v}_i=\mathbf{v}$. To leading order in the ratio of electron to ion mass, the electron velocity, $\mathbf{v}_e=\mathbf{v}-\mathbf{j}/ne$, is expressed in terms of $\mathbf{v}$ and current density $\mathbf{j}$ given by the Ampère's law,

$$\mathbf{j} = \frac{c}{4\pi}\nabla\times\mathbf{B}. \tag{3}$$

The derivative $d\mathbf{v}_e/dt$ on the rhs of (2) is presented as a function of $\mathbf{v}$ and $\mathbf{j}$,

$$\frac{d\mathbf{v}_e}{dt} = \frac{d\mathbf{v}}{dt}-(\mathbf{v}\nabla)\frac{\mathbf{j}}{ne}-\left(\frac{\mathbf{j}}{ne}\nabla\right)\mathbf{v}-\frac{\partial}{\partial t}\frac{\mathbf{j}}{ne}+\left(\frac{\mathbf{j}}{ne}\nabla\right)\frac{\mathbf{j}}{ne}. \tag{4}$$

Although (4) is multiplied by electron mass m_e and, therefore, makes a small contribution to (2), it plays an important role in high-temperature plasma, providing the mechanism of collisionless reconnection. As it is shown in Appendix A, the partial time derivative is the dominant term in (4) and, therefore, we take

$$\frac{d\mathbf{v}_e}{dt} \simeq -\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{ne}\right). \tag{5}$$

Fluid motion is driven by the magnetic and plasma pressure gradients. Variations of ion and electron pressures are assumed to be of the form

$$\frac{\partial p_{i,e}}{\partial t}+(\mathbf{v}_{i,e}\cdot\nabla)p_{i,e}+\gamma_{i,e}p_{i,e}\nabla\cdot\mathbf{v}_{i,e}=0, \tag{6}$$

where $\gamma_i=\gamma_e=5/3$ for an adiabatic equation of state or $\gamma_i=\gamma_e=1$ if the plasma is isothermal.

Under the condition of quasineutrality $n_e=n_i=n$, electron and ion continuity equations are identical to each other and expressed by the plasma continuity equation

$$\frac{\partial n}{\partial t}+\nabla\cdot n\mathbf{v}=0. \tag{7}$$

In order to close the equations we use Faraday's induction equation

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \tag{8}$$

In our scheme of calculations, the functions $\mathbf{v}$ and $\mathbf{j}$ in (2) are presented in terms of $\mathbf{B}$ from (1) and (3) and the resulting expression for $\mathbf{E}$ is substituted in (8), forming equations for two components of $\mathbf{B}$ (the third component is calculated from $\nabla\cdot\mathbf{B}=0$). This procedure is equivalent to the approach based on vector potential presentation for the electric field,

$$\mathbf{E} = -\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{9}$$

where the perpendicular components of the magnetic field are expressed as a function of $\mathbf{A}_\parallel$, or, equivalently, in terms of the flux function ψ.

We restrict our consideration to the simplest fluid equations by ignoring ion gyroviscosity in (1) and (6). Within the scope of this approach, the ion-sound gyroradius ρ_s appears in the final equations due to coupling of the Hall term and the effect of plasma compressibility in a maner similar to that described in Ref. 15. However, we allow for arbitrarily large β and take into account the effect of B_z diffusion. In the hot plasmas limit, our equations are also similar to the linearized version of the four-field model investigated in Ref. 13 and give the results that are in reasonable agreement with corresponding kinetic treatments.[6,17] In these calculations, two layer scales are accounted for; an outer layer where ion motion is important and an inner region where electron inertia and/or resistivity is accounted for. Good agreement between the two-fluid theory and kinetic calculations when ρ_s is larger than δ is due to the fact that in the inner layer the electrons are decoupled from the ions and hence the slow ion motion can be ignored. In the outer zone, the length scales of the solutions are larger than the ion-sound gyroradius; hence a fluid theory approach is applicable.

Since the radial width of the tearing solutions is much shorter than the plasma radius, one can treat the layer problem in slab geometry. We introduce an orthogonal coordinate system with $x=r-r_s$ oriented in the direction in which equilibrium quantitatives vary and y, z oriented along sheared and guiding components of the unperturbed magnetic field, respectively. The origin of the reference frame, $x=0$, is placed at the resonant surface [the equivalent of $q(r_s)=m/n$ for a magnetically confined device]. For example, in the case of the reversed-field pinch (RFP), the $m=0$ mode is resonant on the resonant reversal surface $q(r_s)=m/n=0$. Then, the poloidal magnetic field $\mathbf{B}_P$ represents a guiding magnetic field while toroidal field $\mathbf{B}_T$ corresponds to the sheared component.

In the equilibrium state, the plasma is at rest, $\mathbf{v}^{(0)}=0$, with uniform and constant density $n^{(0)}$, pressures $p_{e,i}^{(0)}$, and total pressure $p^{(0)}=p_e^{(0)}+p_i^{(0)}$. This force-free configuration, $\mathbf{j}^{(0)}\times\mathbf{B}^{(0)}=0$, is described by the equilibrium magnetic field,

$$\mathbf{B}^{(0)}(x)=\mathbf{e}_z B_z^{(0)}(x)+\mathbf{e}_y B_y^{(0)}(x), \tag{10}$$

where $B_z^{(0)}(x)$ is the guiding magnetic field and $B_y^{(0)}(x)$ is the sheared magnetic field. As an example of the x dependence of $B_y^{(0)}(x)$, we will use the sheet pinch profile

$$B_y^{(0)}(x)=B_y^{(\infty)}\tanh\frac{x}{L}, \tag{11}$$

however, the results obtained can be easily modified for other cases. The force-free equilibrium condition $B_y^{(0)2}(x)+B_z^{(0)2}(x)=B_0^2=\text{const}$ leads to the dependence of $B_z^{(0)}$ on x,

$$\frac{dB_z^{(0)}}{dx}=-\frac{B_y^{(0)}(x)}{B_z^{(0)}}\frac{dB_y^{(0)}}{dx}, \tag{12}$$

and, correspondingly, the equilibrium current is given by

$$\mathbf{j}^{(0)}(x)=\left(\mathbf{e}_z+\mathbf{e}_y\frac{B_y^{(0)}(x)}{B_z^{(0)}}\right)\frac{c}{4\pi}\frac{dB_y^{(0)}}{dx}. \tag{13}$$

We will characterize the value of the sheared component $B_y^{(0)}(x)$ by the ratio $\epsilon=B_y^{(\infty)}/B_z^{(0)}(0)$. In accordance with the equilibrium condition, ϵ encompasses the range $0\leqslant\epsilon\leqslant 1$. The basic equations are derived in Appendices A and B for the general case of arbitrary ϵ and dependences $B_y^{(0)}(x)$. However, in the case of practical interest guiding field $B_z^{(0)}$ is strong, $\epsilon\ll 1$, and, correspondingly, the dependence $B_z^{(0)}(x)$ is weak so that we will neglect small variations of $B_z^{(0)}$ and treat it as a constant. For the same reasons one can ignore the small sheared component of the equilibrium current $j_y^{(0)}(x)$ assuming that the unperturbed current is oriented in the z direction. The corresponding simplifications are done in the final equations (15) and (16) by ignoring the terms of order $O(\epsilon)$.

Using the above equilibrium profiles the equations (1)–(7) are linearized with respect to small perturbations of the form

$$\mathbf{B}(x,y,t)=\mathbf{B}^{(0)}(x)+\mathbf{B}(x)\exp(-i\omega t+iky),$$
$$\mathbf{v}(x,y,t)=\mathbf{v}(x)\exp(-i\omega t+iky). \tag{14}$$

The vector $\mathbf{k}=k\mathbf{e}_y$ is oriented along y to satisfy resonant condition $F(0)=0$ at $x=0$, where $F(x)=\mathbf{k}\cdot\mathbf{B}^{(0)}(x)$, and all perturbations are uniform in the z direction $(\partial/\partial z=0)$. After transformations decribed in detail in (A3)–(A7), one obtains equations for the perturbed quantities B_x and B_z,

$$-i\omega(B_x-\delta^2\nabla^2 B_x)=(ikv_x+\alpha_{\mathrm{H}}k^2B_z)B_y^{(0)}, \tag{15}$$

$$i\omega\left[\left(1-\frac{k^2v_a^2(x)}{\omega^2}\right)B_z-\delta^2\boldsymbol{\nabla}^2B_z\right]$$
$$=B_z^{(0)}\,\boldsymbol{\nabla}\cdot\mathbf{v}-\alpha_{\mathrm{H}}\left(B_y^{(0)}(x)\boldsymbol{\nabla}^2B_x-\frac{d^2B_y^{(0)}}{dx^2}B_x\right), \tag{16}$$

where $\boldsymbol{\nabla}^2=d^2/dx^2-k^2$, the Hall constant is defined by $\alpha_{\mathrm{H}}=c/(4\pi e n^{(0)})$, $\delta^2=d_e^2+ic^2\eta/(4\pi\omega)$ describes the combined collisionless and collisional skin depth, $d_{e,i}=c/\omega_{pe,i}$, $\omega_{pe,i}^2=4\pi e^2 n^{(0)}/m_{e,i}$. The term proportional to $v_a^2(x)=B_y^{(0)2}(x)/(4\pi\rho^{(0)})$ originates from $ikv_zB_y^{(0)}$ in (A7), where v_z is expressed in terms of B_z from (B7). Following Ref. 22, we use Eq. (16) to accurately treat plasma compressibility $\boldsymbol{\nabla}\cdot\mathbf{v}$ in order to derive the universal set of equations applicable for arbitrary β.

Coupling between magnetic field and fluid motion results from the velocity-dependent terms in (15) and (16) proportional to v_x and $\boldsymbol{\nabla}\cdot\mathbf{v}$. We use Eq. (16) to eliminate terms proportional to $\boldsymbol{\nabla}\cdot\mathbf{v}$ in subsequent equations. This allows us to form a closed set of equations for v_x, B_x, and B_z. These equations represent a generalization of the resistive MHD equations for the case of two-fluid theory where magnetic perturbations along the guiding field B_z are important. Instead of using equations of motion for v_x and v_y [see (B5) and (B6)], we find it convenient to solve equations for $\nabla^2 v_x$ (the vorticity equation) and plasma compressibility $\boldsymbol{\nabla}\cdot\mathbf{v}$. Following calculations outlined in Appendix B, the desired equations are given by

$$\frac{-\omega}{k}\left(\nabla^2 v_x-\frac{d}{dx}\,\boldsymbol{\nabla}\cdot\mathbf{v}\right)$$
$$=\frac{1}{4\pi\rho^{(0)}}\left(B_y^{(0)}(x)\boldsymbol{\nabla}^2B_x-\frac{d^2B_y^{(0)}}{dx^2}B_x\right), \tag{17}$$

$$-i\omega\left(1+\frac{c_s^2}{\omega^2}\nabla^2\right)\boldsymbol{\nabla}\cdot\mathbf{v}$$
$$=-\frac{1}{4\pi\rho^{(0)}}\nabla^2\left(B_z^{(0)}B_z+\frac{iB_y^{(0)}(x)}{k}\frac{dB_x}{dx}\right)$$
$$+\frac{ik}{2\pi\rho^{(0)}}\frac{dB_y^{(0)}}{dx}B_x, \tag{18}$$

where the ion sound speed is given by $c_s^2=(\gamma_eT_e^{(0)}+\gamma_iT_i^{(0)})/m_i$. We use (16) to substitute for the plasma compressibility $\boldsymbol{\nabla}\cdot\mathbf{v}$ into (17) and (18). The substitution of $\boldsymbol{\nabla}\cdot\mathbf{v}$ into (17) yields the vorticity equation with the effect of plasma compressibility taken into account. The substitution into (18) leads to a basic differential equation that does not depend on $\mathbf{v}$ and describes the relationship between B_z and B_x,

$$\left(1+\frac{c_s^2}{\omega^2}\boldsymbol{\nabla}^2\right)\left[\frac{\omega^2}{v_{\mathrm{A}}^2}\left(B_z\left(1-\frac{k^2v_a^2(x)}{\omega^2}\right)-\delta^2\boldsymbol{\nabla}^2B_z\right)-\frac{i\omega\alpha_{\mathrm{H}}}{v_{\mathrm{A}}^2}\left(B_y^{(0)}\right.\right.$$
$$\left.\left.\times(x)\boldsymbol{\nabla}^2B_x-\frac{d^2B_y^{(0)}}{dx^2}B_x\right)\right]$$
$$=-\nabla^2\left(B_z+\frac{iB_y^{(0)}(x)}{kB_z^{(0)}}\frac{dB_x}{dx}\right)+\frac{2ik}{B_z^{(0)}}\frac{dB_y^{(0)}}{dx}B_x, \tag{19}$$

where $v_A = B_z^{(0)}/(4\pi\rho^{(0)})^{1/2}$. Then, Eqs. (15)–(17) and (19) constitute a closed set of equations for B_x, B_z, and v_x that describe the general case of two-fluid tearing instability.

In Eq. (19), the effect of plasma compressibility is essential. This equation will be simplified in two limiting cases of small and large values of $(c_s^2/\omega^2)\nabla^2$. Applying (19) to the case of slow growing tearing modes with $L\omega/c_s \ll 1$ (hot plasma limit), yields $1+(c_s^2/\omega^2)\nabla^2 \to (c_s^2/\omega^2)\nabla^2$. We solve the resulting equation using boundary conditions at $x\to\pm\infty$, and take into account the tearing parity of eigenfunctions; $B_x(x)$ is even in x and $B_z(x)$ is an odd function of x, $dB_x/dx(0)=0$ and $B_z(0)=0$. The last two terms on the rhs of (19) violate this symmetry, showing that the general solution has no parity. Since these terms are order $\epsilon \ll 1$, they can create only small perturbations of the opposite parity so that the resulting contribution from these corrections to (19) is proportional to ϵ^2 and, therefore, can be neglected. Solving the resulting equation is equivalent to solving an equation of the form $\nabla^2 f = (d^2/dx^2 - k^2)f = 0$. Noting that f is odd, the solution is $f(x) = C(|x|/x)\exp(-k|x|)$. Using continuity at $x=0$ requires $C=0$, or, equivalently, $f=0$. Thus, the leading-order solution to (19) is

$$\frac{v_A^2}{c_s^2}B_z + B_z\left(1 - \frac{k^2 v_a^2(x)}{\omega^2}\right) - \delta^2\nabla^2 B_z - \frac{i\alpha_H}{\omega}\left(B_y^{(0)}(x)\nabla^2 B_x - \frac{d^2 B_y^{(0)}}{dx^2}B_x\right) = 0. \tag{20}$$

The guiding field $B_z^{(0)}$ enters (20) via the factor $(1+\beta^{-1})B_z$, where β is defined as

$$\beta = \frac{c_s^2}{v_A^2}. \tag{21}$$

This differs slightly from the usual definition $\beta = 8\pi p/B_z^{(0)2}$ by the factor $\gamma_{e,i}/2$.

Equation (20) describes quasistatic equilibration of magnetic and thermal pressures, $B_z^{(0)}B_z/4\pi + p = 0$, where p is expressed in terms of $\nabla\cdot\mathbf{v}$, which, in turn, is calculated from the induction equation (16). A small part of the magnetic pressure, $B_y^{(0)}B_y/4\pi$, is proportional to ϵ^2 and, therefore, ignored due to the arguments mentioned above. As it is shown in Sec. III, Eq. (20) describes two-fluid kinetic Alfvén and magneto-acoustic waves while the compressional Alfvén wave is decoupled. We refer to this regime as the hot plasma case, or kinetic Alfvén-driven tearing instability

In the opposite limit of cold plasma, one can neglect $1+(c_s^2/\omega^2)\nabla^2 \to 1$ on the lhs of (19) and keep the term $\nabla^2 B_z$ on the rhs of this equation, yielding another relationship,

$$\frac{\omega^2}{v_A^2}(B_z - \delta^2\nabla^2 B_z) + \nabla^2 B_z = \frac{i\omega\alpha_H}{v_A^2}\left(B_y^{(0)}(x)\nabla^2 B_x - \frac{d^2 B_y^{(0)}}{dx^2}B_x\right), \tag{22}$$

where a small contribution from coupling with the magneto-acoustic wave, $k^2 v_a^2(x)/\omega^2$, is ignored. Equation (22) describes a nonvanishing B_z as $\beta \to 0$. In this case, magnetic pressure $B_z^{(0)}B_z/4\pi$ is balanced by the ion inertia as it is for the compressional Alfvén wave. Since the CA mode exhibits the properties of whistlers for $k_\parallel \neq 0$ and $kd_i \gg 1$, we will refer to this regime as the cold plasma case, or whistler-mediated tearing mode.

The terms with plasma compressibility $\nabla\cdot\mathbf{v}$ are present in the induction and vorticity equations (16) and (17). If the guiding magnetic field is strong, plasma flow can be treated as incompressible in the vorticity equation (17). This is motivated by noting from (16) that $\nabla\cdot\mathbf{v} \propto 1/B_z^{(0)}$, and, correspondingly, the incompressible approximation is applicable at a strong enough guiding field. Making use of (20) and (22) allows us to present $\nabla\cdot\mathbf{v}$ in two forms related to the hot and cold plasma approximations, respectively,

$$\nabla\cdot\mathbf{v} = -i\omega\frac{v_A^2}{c_s^2}\frac{B_z}{B_z^{(0)}}, \quad \nabla\cdot\mathbf{v} = -i\frac{v_A^2}{\omega}\nabla^2\frac{B_z}{B_z^{(0)}}. \tag{23}$$

The magnitude of B_z in (23) depends on the approximation considered. For example, in the cold plasma case with small Δ', the eigenfunction B_z is determined by the balance of the rhs of (22) and the term $\nabla^2 B_z$ on the lhs of this equation. In the hot plasma case with relatively small β, the terms proportional to B_z dominates on the lhs of (20). Requiring the smallness of $d/dx\nabla\cdot\mathbf{v}$ in comparison with the rhs of (17) yields a universal condition for the incompressible approximation in the vorticity equation

$$\frac{d}{dx} \ll k(\beta+1)\frac{\omega_{Bi}}{\omega}. \tag{24}$$

Applying (24) for the hot plasma case shows that this approximation is valid at the relatively weak restriction $\epsilon \ll L\delta/(\rho_s d_i)$. In the cold plasma limit, this condition is more rigorous so that the effect of compressibility requires more a detailed analysis, which will be presented in a separate paper.

In the induction equation (16), the small term $\nabla\cdot\mathbf{v}$ is multiplied by the large factor $B_z^{(0)}$, so that the the applicability of an incompressible approximation in (16) requires an additional analysis. For this purpose, it is suitable to use (19) from which the unknown term $\nabla\cdot\mathbf{v}$ is excluded. At $\beta = c_s^2/v_A^2 \gg 1 \gg \omega^2 L^2/v_A^2$, Eq. (19) is equivalent to (20), where the first term, B_z/β, on the lhs is ignored. The resulting equation coincides with the incompressible version of (16) (with $\nabla\cdot\mathbf{v}=0$). This leads to the well-known conclusion[22,26] that the incompressible approximation in the induction equation is valid at $\beta \gg 1$. We will show in Sec. V that the area of applicability of an incompressible approximation in the induction equation is wider. Indeed, at small enough Δ', $\Delta'\delta \ll \sqrt{\beta} \ll 1$, the tearing layer is so narrow that the term $\delta^2\nabla^2 B_z$ dominates on the lhs of (16) making the effect of plasma compressibility unimportant although $\beta \ll 1$ in this case. At smaller β, $\sqrt{\beta} \ll \Delta'\delta$, the term $-B_z^{(0)}\nabla\cdot\mathbf{v}$ becomes important. It compensates all other terms on the rhs of (16), and a residual effect determines a small $B_z \propto \beta$. A universal equation valid at arbitrary β is given by (20).

III. LOW-FREQUENCY WAVES IN UNIFORM MAGNETIC FIELD

Different regimes of two-fluid tearing instability described by (15)–(17) and (19) can be classified by applying these equations to the case of plane waves propagating in a uniform magnetic field. Without loss of generality, we will use the same geometry $\mathbf{k}=(0,k,0)$ and $\mathbf{B}^{(0)}=(0,B_y^{(\infty)},B_z^{(0)})$. Introducing an angle θ between $\mathbf{k}$ and $\mathbf{B}^{(0)}$ yields $B_z^{(0)}=B^{(0)}\sin\theta, B_y^{(0)}=B^{(0)}\cos\theta$. Since $\mathbf{k}$ is oriented along y, one can drop x derivatives in (15)–(17) and (19). Substituting B_z and v_x from (19), (17), and (15) gives a dispersion equation that determines $\omega(k,\theta)$. Similar two-fluid dispersion relations were analyzed in many publications (see, for example, Ref. 23). A suitable form (25) is found below by introducing an effective phase velocity $V(k)=\omega(1+k^2d_e^2)^{1/2}/kv_A$ and an effective beta $\beta_k=(1+k^2d_e^2)\beta$, where $v_A=B^{(0)}/(4\pi\rho^{(0)})^{1/2}$. Then, the variables V and k are separated so that the lhs of (25) depends on V while the rhs is a function of k,

$$(V^2-\cos^2\theta)\left(1-\frac{\cos^2\theta}{V^2}-\frac{\sin^2\theta}{V^2-\beta_k}\right)=\frac{k^2d_i^2\cos^2\theta}{(1+k^2d_e^2)}. \quad (25)$$

Rigorously speaking, the lhs of (25) depends also on k via the factor β_k. For the range $kd_e\lesssim 1$, this dependence is weak and can be ignored by putting, $\beta_k\simeq\beta$. This is assumed in the ensuing discussion. However, this effect is important and will be taken into account in Sec. V for tearing modes at small Δ'.

Plotting the lhs of (25) as a function of V and intersecting it with the horizontal line that corresponds to the Hall term on the rhs yields three branches of oscillations illustrated in Fig. 1. At small $k\to 0$, solutions $V(k)$ tend to their limiting values $V(0)$, which correspond to phase velocities of single-fluid MHD waves; shear Alfvén (SA), compressional Alfvén (CA), and magneto-acoustic (MA). These values are given by the zeros of the lhs of (25). At large k, the modes are coupled and modified due to the presence of the Hall term. We will label the modified branches in accordance with the above classification in a single-fluid MHD limit at $k\to 0$.

The slopes of the curves in Fig. 1 depend on interrelation between $\cos^2\theta$ and β_k. Figure 1(a) illustrates the case $\beta_k<\cos^2\theta$ when phase velocities of the MA and the SA waves decrease with the increase of k. In this case, the maximum of $V_{\rm MA}$ and $V_{\rm SA}$ are achieved at small k when two-fluid effects are unimportant.

In contrast to this, at $\beta_k>\cos^2\theta$ the SA curve changes its slope and the SA phase velocity becomes a growing function of k [see Fig. 1(b)]. Starting from $V_{\rm SA}(0)=\cos\theta$ at $k=0$ the phase velocity $V_{\rm SA}(k)$ tends to β_k when $k\to\infty$. In the intermediate range $\cos^2\theta\ll V^2\ll\beta_k$, an analytic solution for the SA branch is obtained by simplifying $V^2-\beta_k\to-\beta_k$ and dropping the term $\cos^2\theta/V^2\ll 1$,

$$\omega_{\rm SA}^2=\frac{k^2v_A^2\cos^2\theta}{1+k^2d_e^2}\left(1+\frac{k^2d_i^2}{1+k^2d_e^2}\frac{\beta_k}{1+\beta_k}\right). \quad (26)$$

At small $\beta_k\ll 1$, Eq. (26) represents the two-fluid MHD analog of the kinetic Alfvén wave with $\omega\simeq k_\parallel v_A\rho_s k$. The ion-sound Larmor radius, $\rho_s=d_i\sqrt{\beta}$, appears in (26) due to the Hall term and plasma compressibility. The appearance of ρ_s is often associated with the electron pressure gradient term in generalized Ohm's law (2). In our case of force free equilibrium, the electron pressure gradient term does not contribute to the induction equation due to the identity $\nabla\times\nabla p_e/n^{(0)}\equiv 0$.

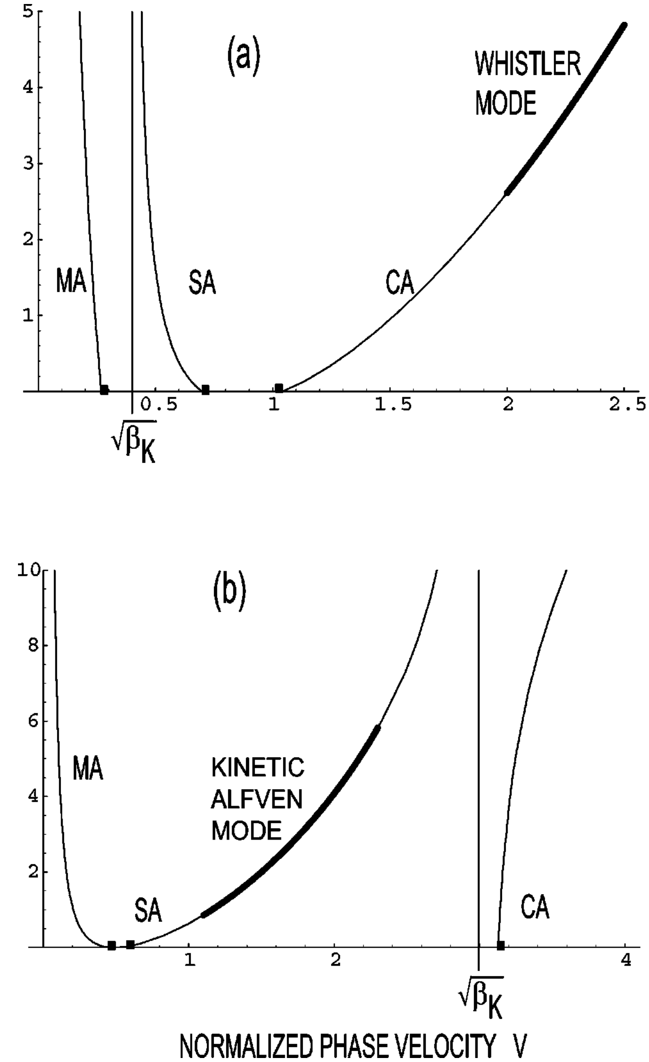

FIG. 1. Dependence of the lhs of the two-fluid dispersion relation (25) on the phase velocity V. Intersections with the horizontal line corresponding to the Hall term on the rhs of this equation yield three branches of oscillations [the magnetoacoustic (MA), shear (SA), and compressional (CA) Alfvén modes] modified on short scales by the two-fluid effects. (a) The case of $\beta_k=\beta(1+k^2d_e^2)<\cos^2\theta$. (b) The case $\beta_k>\cos^2\theta$.

At large $\beta_k\gg 1$, the second term in (26) does not depend on β_k and represents a pure electron response with the whistler dispersion $\omega\propto k_\parallel v_A d_i k$. We will show in Sec. V that if Δ' is large and the tearing mode is localized on a wide scale $\simeq\rho_s$, then starting from some critical β_c, the effect of coupling with the MA wave slows down the instability. At small Δ', when the tearing mode is localized on a short scale, the effect of the MA wave is unimportant, and the whistler-mediated reconnection can be achieved at the soft conditions $\beta\ll 1$ and $d_i\gg\delta$. This is possible due to the dependence of β_k on k that can provide $\beta_k\gg 1$ at $\beta\ll 1$.

As it is seen in Fig. 1, the phase velocity $V(k)$ of the MA wave decreases with the increase of k. An asymptotic solution for this low-frequency mode is obtained from (25), simplified at $V^2\ll\beta_k,\cos^2\theta$. This yields the standard MHD expression for the frequency of the MA wave modified on short scales due to two-fluid effects,

$$\omega_{\rm MA}^2 = \frac{k^2\cos^2\theta c_s^2}{1+2\beta+k^2\rho_s^2}. \tag{27}$$

The asymptotic solution for high-frequency CA wave is obtained by neglecting $\cos^2\theta$ in the first brackets on the rhs of (25) and β_k in the denominator in the second brackets ($V^2 \gg \cos^2\theta, \beta_k$),

$$\omega_{\rm CA}^2 = \frac{k^2 v_{\rm A}^2}{1+k^2 d_e^2}\left(1+\frac{k^2 d_i^2\cos^2\theta}{1+k^2 d_e^2}\right). \tag{28}$$

Similar to (26) at $\beta_k \gg 1$, the second term in (28) describes the pure electron response with the typical whistler dependence $\omega \simeq k_\parallel v_{\rm A} d_i k$. In contrast to the SA mode, this response within the scope of the CA mode does not require $\beta_k \gg 1$ and can be achieved even at $\beta_k = 0$. However, the frequency of whistlers tends to zero on the resonant surface, $k_\parallel = 0$, causing this branch to exhibit the properties of the CA mode with non zero frequency $\omega = k v_{\rm A}$ determined by the guiding field and ion mass. This suppresses the out-of-plane perturbation B_z and slows down the instability. In order to realize the conditions of whistler-mediated reconnection, d_i should be much greater than the scale of equilibrium magnetic field L; otherwise the instability develops in the regime of single-fluid MHD.

IV. TWO-FLUID TEARING EQUATIONS

A. Instability driven by the SA and the CA modes (cold plasma case, $\beta = 0$)

The set of tearing equations related to the cold plasma case, $\gamma l / \gg c_s$ (l is the shortest spatial scale) consists of Eq. (15), (17), and (22). Introducing the growth rate of the pure growing tearing mode $\gamma = -i\omega$ and dimensionless variables $v = i v_x / v_a$, $B_{x,z} \to B_{x,z}/B_y^{(\infty)}$, $\tau_a = L/v_a$, $x \to x/L$, $d_{e,i} \to d_{e,i}/L$, $\delta \to \delta/L$, $k \to kL$, $B_y^{(0)}(x) \to B_y^{(0)}(x)/B_y^{(\infty)} = \tanh x$ where $v_a = B_y^{(\infty)}/(4\pi\rho^{(0)})^{1/2}$, yields these equations in the form

$$B_z - \left(\delta^2 + \frac{1}{\gamma^2\tau_a^2\epsilon^2}\right)\nabla^2 B_z = \frac{d_i}{\gamma\tau_a}\left(B_y^{(0)}(x)\nabla^2 B_x - \frac{d^2 B_y^{(0)}}{dx^2}B_x\right), \tag{29}$$

$$\frac{\gamma\tau_a}{k}(B_x - \delta^2\nabla^2 B_x) = (v + d_i k B_z)B_y^{(0)}(x), \tag{30}$$

$$\frac{\gamma\tau_a}{k}\nabla^2 v = \left(\frac{d^2 B_y^{(0)}}{dx^2}B_x - B_y^{(0)}(x)\nabla^2 B_x\right), \tag{31}$$

where, in accordance with (24), the effect of plasma compressibility is ignored in the vorticity equation (17). The system (29)–(31) coincides with the equations derived with the use of a different approach in Ref. 11, where the transition from the single fluid to the Hall MHD was investigated within the scope of the constant-ψ approximation.

Applying Eq. (29)–(31) for the case of waves propagating in a uniform magnetic field yields a dispersion relation that is identical to (25), but with $\beta_k = 0$. This corresponds to the MA mode being decoupled. Hence, it does not play a role in the reconnection layer. Within the scope of (29)–(31) the tearing instability is driven by the SA and the CA modes modified on short scales by the the Hall term. A short-wavelength continuation of the CA mode represents a pure electron whistler mode (28), which can drive a fast whistler mediated tearing instability.

Electron MHD (EMHD) whistler tearing equations are derived from (29)–(31) by making use of the formal limiting transition $m_i \to \infty$. Ion mass-dependent parameters $d_i \propto \sqrt{m_i}$ and $\tau_a \propto \sqrt{m_i}$ determine the ordering of this transition, while the growth rate is independent of the ion mass. Considering (31) at $\tau_a \to \infty$ and keeping γ=const, yields $v \propto \tau_a^{-1} \to 0$. Putting $v = 0$ in (30) gives Eq. (33), while Eq. (32) for B_z is obtained by dropping the term proportional to $1/(\gamma\tau_a\epsilon)^2$ on the lhs of (29),

$$B_z - \delta^2\nabla^2 B_z = \frac{d_i}{\gamma\tau_a}\left(B_y^{(0)}(x)\nabla^2 B_x - \frac{d^2 B_y^{(0)}}{dx^2}B_x\right), \tag{32}$$

$$B_x - \delta^2\nabla^2 B_x = \frac{d_i k^2}{\gamma\tau_a}B_y^{(0)}(x)B_z. \tag{33}$$

Equations (32) and (33) are written in the form similar to Ref. 10 and do not depend on ion mass. They represent the whistler-driven tearing instability first considered in Ref. 7. In accordance with Ref. 8, the growth rate scales at small Δ' as $\gamma_w \tau_a / k \propto d_i(\delta\Delta')^2$. The large term $\propto 1/(\gamma\tau_a\epsilon)^2$ in (29) prevents accessibility to the whistler regime of instability in plasma with a strong guiding field and small d_i. This reflects the effect mentioned in Sec. III when ion motion in the compressional Alfvén wave reduces B_z in the vicinity of the resonant surface $x = 0$ and turns the instability into the single-fluid MHD regime. To overcome this barrier, the tearing instability should be fast enough, $\gamma\tau_a\epsilon \gg \delta^{-1}$. Substituting γ_w in this inequality shows that d_i has to be much greater than the external scale L, $c/\omega_{pi} \gg (m_i/m_e)^{1/4} L/\epsilon^{1/2}$. Generally this is not satisfied in magnetically confined plasmas. Note that the tearing equations and, correspondingly, the growth rate does not depend on ϵ in both limiting cases of single-fluid MHD and a pure electron whistler mode (32) and (33). However, this dependence is present in (29)–(31), and can, therefore, be important in the transition between the regimes.

B. Instability driven by the SA and the MA modes (hot plasma case)

The set of tearing equations in the hot plasma case, $c_s \gg \gamma L$, consists of Eqs. (15), (17), and (20). We will deal with the dimensionless version of these equations (see Sec. IV A), where Eqs. (15) and (17) are represented by (30), (31) while Eq. (20) takes the form

$$B_z\left(1 + \frac{1}{\beta} + \frac{k^2 B_y^{(0)}(x)^2}{\gamma^2\tau_a^2}\right) - \delta^2\nabla^2 B_z = \frac{d_i}{\gamma\tau_a}\left(B_y^{(0)}(x)\nabla^2 B_x - \frac{d^2 B_y^{(0)}}{dx^2}B_x\right). \tag{34}$$

Equation (34) corresponds to the quasistatic regime of tearing instability when the total (magnetic + thermal) pressure is equilibrated across the layer. The term B_z/β originates from the term with plasma compressibility in the induction equation (16). As β grows, Eq. (34) describes the transition from compressible to incompressible versions of this equation. Dropping the term $\delta^2\nabla^2 B_z$ responsible for the diffusion of B_z and simplifying $1+1/\beta \simeq 1/\beta$ at $\beta \ll 1$, the system (30), (31), and (34) coincides with the equations obtained in Ref. 13 from the four-field model.

Applying (30), (31), and (34) for plane waves propagating in a uniform magnetic field shows that in the hot plasma case, the instability is driven by the coupling of the SA (26) and the MA (27) modes while the CA mode (28) is decoupled and plays no role in the reconnection layer. The main driving force of the instability is the SA mode (26), which is a two-fluid analog of the kinetic Alfvén wave. As $\beta_k \to \infty$ (at $kd_e \to \infty$), the SA mode exhibits dispersion properties of whistlers so that one may expect a whistler-like behavior for the tearing eigenmodes.

Making limiting transitions $\beta \to \infty$ and $m_i \to \infty$ with $\tau_a \propto d_i \propto \sqrt{m_i} \to \infty$ allows us to drop the v-dependent term in (30) and the terms B_z/β and $k^2 B_y^{(0)2}/\gamma^2\tau_a^2$ in (34). After these transformations equations (30), (31), and (34) reduce to the EMHD whistler equations (32) and (33). Using its solutions[10] at small $\Delta'\delta \ll 1$ yields the conditions of validity of the above transformations. According to Ref. 10, the width of the tearing layer is $\delta^2\Delta'$, so that the $\delta^2 d^2 B_z/dx^2$ term dominates on the lhs of (34). Taking this into account, equating the rhs of (34) and (31), and integrating twice over x yields

$$B_z = (d_i/\delta^2 k)v. \tag{35}$$

Comparing the terms v and $kd_i B_z$ in (30) shows that the $v=0$ approximation is valid when $d_i \gg \delta$. This criterion also allows us to drop the MA wave term relative to B_z diffusion in (34). The term B_z/β in (34) can also be dropped if $\Delta'\delta \ll \sqrt{\beta}$. Therefore, at small $\beta \ll 1$, the whistler-mediated regime of instability can be achieved. Indeed, according to Sec. III the effective beta, $\beta_k = (1+k^2 d_e^2)\beta$, determines the dispersion properties of the modes. If the instability is localized on short scale, $kd_e \gg 1$, then β_k can be large, $\beta_k \gg 1$, even if the usual beta is small, $\beta \ll 1$. This is caused by the diffusive term in (34), which dominates at $\Delta'\delta \ll \sqrt{\beta} \ll 1$. When Δ' increases and exceeds this limit, the tearing mode transitions from the whistler regime to the kinetic Alfvén regime with the growth rate γ determined by ρ_s instead of d_i. Correspondingly, the first term dominates on the lhs of (34), yielding solution (46). In all cases of small Δ', the coupling with the MA wave is small.

At larger Δ', the eigenfunctions are broadened in its spatial width to the order of ρ_s and, correspondingly, coupling with the MA wave becomes significant. To illustrate this, we will solve in Sec. V Eqs. (30), (31), and (34) in a wide range of Δ' and large enough β (but still $\beta \ll 1$) when coupling with the MA wave effects B_z. The range of β considered and the method of calculations are similar to those treated in Ref. 13. However, our equations contain the important term of B_z diffusion. This effect was not included in Ref. 13.

V. TWO-FLUID TEARING INSTABILITY IN HOT PLASMA REGIME

The above analysis describes the transition from the whistler to the kinetic Alfvén regime in terms of the stability factor Δ'. We now introduce three possible regimes that reflects the dependence on the guiding magnetic field.

At small beta, $\beta \ll m_e/m_i$ (regime A), ion-sound gyroradius $\rho_s \ll \delta$ and, correspondingly, the electron and ion flows are coupled on the tearing layer scale δ, yielding the single-fluid MHD regime of tearing instability. In the case of small $\Delta'\delta \ll 1$, the width of the tearing layer is $\Delta'\delta^2 \ll \delta$ and, correspondingly, the condition of single-fluid MHD becomes stronger.

At finite beta, $m_e/m_i \ll \beta \ll \beta_c = (\delta/d_i)^{1/2} \gtrsim (m_e/m_i)^{1/4}$ (regime B), the ion-sound gyroradius exeeds the electron skin depth, $\rho_s \gg \delta$ (this particular expression for β_c corresponds to the case of large Δ'). Electrons and ions are decoupled on a spatial scale smaller than the ion-sound gyroradius, however, effects of the MA wave coupling and B_z diffusion are still unimportant in (34). The tearing layer consists of two sublayers: a narrow diffusive layer of the width $\simeq \Gamma\delta$, where electron diffusivity is important, and a layer specific to two-fluid theory $\Gamma\delta \ll |x| \lesssim \rho_s$ with ideal flows of decoupled electron and ions.[6,15] Here we use a normalized growth rate,

$$\Gamma = \frac{\gamma\tau_a}{\rho_s k}. \tag{36}$$

At small Δ', the value of β_c is determined by $\beta_c \simeq (\Delta'\delta)^2$. The two values for β_c are well matched at $\Delta_c'\delta^{2/3}\rho_s^{1/3} \simeq 1$, or, equivalently, at $\Delta_c'\delta^{3/4}d_i^{1/4} \simeq 1$. The factor Δ_c' is a characteristic value of Δ' at which transition between the limiting cases of small and large Δ' takes place. The value of Δ_c' is approximately equal to δ^{-1} at very small β (in the resistive MHD limit), then gradually decreases as $\Delta_c' \simeq \delta^{-1}(\delta/d_i)^{1/3}\beta^{-1/6}$ in the range $m_e/m_i \ll \beta \ll \beta_c$ and reaches its minimum value $\Delta_c' \simeq \delta^{-1}(\delta/d_i)^{1/4} \simeq \delta^{-1}(m_e/m_i)^{1/8}$ at $\beta \gg \beta_c$. This is just slightly (approximately 2.6 times) smaller than the value $\Delta_c' \simeq \delta^{-1}$ in the single-fluid MHD limit.

At $\beta \gg \beta_c$ (regime C), the effects of the MA wave and B_z diffusion become important. The contribution from the MA wave was calculated in Ref. 13. However, the effect of diffusion of B_z was ignored in this paper. We will show that it plays a significant role and is equally important with the effect of MA wave leading to an additional decrease of the growth rate. Mutual action of these two mechanisms is calculated below giving an universal description that covers both regimes (B) and (C) and the transition between them.

Solving (30), (31), and (34), we appy a boundary layer technique based on asymptotic matching of the tearing inner, $|x| \ll 1$, and ideal outer, $|x| \gg \rho_s$, solutions. The inner layer consists of aforementioned sublayers based on δ, ρ_s, and d_i scales. This layer is extended up to $|x| \ll 1$ so that one can simplify (30), (31), and (34) in this region by Taylor expanding $B_y^{(0)}(x) \to x$. This leads to the internal equations (37)–(39):

$$\left(1 + \frac{\beta x^2}{\rho_s^2\Gamma^2}\right)B_z - \beta\delta^2\frac{d^2 B_z}{dx^2} = \frac{\beta^{1/2}x}{k\Gamma}\frac{d^2 B_x}{dx^2}, \tag{37}$$

$$\rho_s\Gamma\left(B_x-\delta^2\frac{d^2B_x}{dx^2}\right)=x(v+d_ikB_z), \tag{38}$$

$$\rho_s\Gamma\frac{d^2v}{dx^2}=-x\frac{d^2B_x}{dx^2}. \tag{39}$$

On the scale $x \gg \rho_s$, referred below to as an outer zone, electrons and ions are coupled within the scope of an ideal single-fluid MHD. The ideal quasistatic solutions in the outer zone follow from (30), (31), and (34) in the limit $\delta \to 0, \gamma \to 0$. The leading term on the lhs of (34) is B_z/γ. This yields $B_z=0$, while the profile of B_x follows from (31),

$$B_y^{(0)}(x)\nabla^2B_x-\frac{d^2B_y^{(0)}}{dx^2}B_x=0. \tag{40}$$

Equation (40) predicts a singularity of dB_x/dx at $x=0$ that is characterized by the stability factor

$$\Delta'=\left(\frac{1}{B_x}\frac{dB_x}{dx}\right)_{+0}-\left(\frac{1}{B_x}\frac{dB_x}{dx}\right)_{-0}, \tag{41}$$

with $\Delta'>0$ corresponds to the instability. Using the Harris sheet pinch profile yields a eigenmode even in x and decaying at infinity with a discontinuity of dB_x/dx at $x=0$,

$$B_x(x)=C_1\exp(-k|x|)\left(1+\frac{\tanh|x|}{k}\right), \tag{42}$$

for which $\Delta'(k)=2/k-2k$. The asymptotic expansion of (42) at $x \ll 1$ is

$$B_x(x)=C_1+C_2|x|, \tag{43}$$

where the ratio of C_1 and C_2 is determined by $C_2/C_1=\Delta'(k)/2$. The asymptotic boundary conditions for (37)–(39) at $\rho_s \ll |x| \to \infty$ take the form

$$B_x(x)\to C_1\left(1+\frac{\Delta'|x|}{2}\right),\quad B_z\to 0,$$

$$v(x)\to v^{(\infty)}=\rho_s\Gamma C_2,\quad u(x)=\frac{dv}{dx}\to 0, \tag{44}$$

while the boundary conditions at $x=0$ follow from the tearing parity of the eigenfunctions,

$$\frac{dB_x}{dx}(0)=v(0)=B_z(0)=0. \tag{45}$$

VI. INNER LAYER SOLUTIONS

The finite beta regime (B), $m_e/m_i \ll \beta \ll \beta_c$, was analyzed with the use of kinetic[6] and two-fluid MHD[15] approaches. Dropping B_z diffusion and the MA wave terms in (37), yields two alternative expressions for B_z,

$$B_z=\frac{x\beta^{1/2}}{k\Gamma}\frac{d^2B_x}{dx^2}=-\frac{\beta d_i}{k}\frac{du}{dx}. \tag{46}$$

Substituting (46) in (38) with $v=0$ gives the profile of $B_z(x)$ that is described at large Δ' by the two-scale eigenfunction with a narrow peak of the width $\Gamma\delta$ at small x and the bulk of distribution localized on the large ρ_s scale. Since B_z diffusion

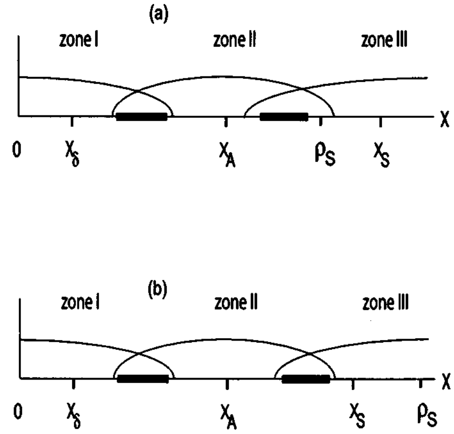

FIG. 2. The three overlapped zones of the internal layer solutions at large Δ'. The thick lines show the intervals where the matching is performed. Case (a) corresponds to the regime (B) with $a=\Gamma/\sqrt{\beta} \gg 1$ when the diffusion of B_z and the MA wave are unimportant; case (b) is related to the regime (C) with $a \ll 1$ when the two above effects are significant. The matching scheme described in Sec. VI covers the subcases (a) and (b) and the transition between them.

and the MA wave terms are proportional to β, these two effects are important at large enough, $\beta \gtrsim \beta_c$. Indeed, the term $\beta x^2B_z/\rho_s^2\Gamma^2$ on the lhs of (37) represents the effect of coupling with the MA wave. It is important at $x>x_s=\rho_s\Gamma/\beta^{1/2}$, when the characteristic frequency of the MA wave is comparable with the growth rate. Since the bulk of the B_z distribution is on the ρ_s scale, the MA wave coupling is important at $x_s<\rho_s$, or, equivalently, at $\beta^{1/2} \gg \Gamma$.

The second term on the lhs of (37) describes diffusion of B_z caused by collisionless and collisional effects included in δ. Generally, the magnetic field diffusion plays a key role in tearing instabilities by providing a mechanism to break the frozen flux theorem and the mode growth on the resonant surface $x=0$. This is provided by the B_x component in (38). The diffusion of B_z is not important for the existence of the instability but it effects the ion–electron decoupling and, thus, on the growth rate of instability. Since the corresponding term in (37) is proportional to $\delta^2\beta \ll 1$, it is significant at small x where the width of localization of B_z is $\Gamma\delta$. The term with the B_z diffusion dominates if $\delta^2\beta d^2B_z/dx^2 \gg B_z$. Estimating $d^2B_z/dx^2 \simeq B_s/(\Gamma\delta)^2$, yields $\beta \gg \Gamma^2$. Thus, the mechanisms of the MA wave on large scale and B_z diffusion on short scale become important when β exceeds a critical value $\beta_c \simeq \Gamma^2 \ll 1$. Using solutions $\Gamma \simeq (\delta/\rho_s)^{1/3}$ at large Δ' and $\Gamma \simeq \Delta'\delta$ at small Δ' from Ref. 6, gives $\beta_c=(\delta/d_i)^{1/2}$ and $\beta_c=(\Delta'\delta)^2$, respectively.

Analytical calculations at $\beta \ll 1$ are simplified due to the fact that the effects of the MA wave and B_z diffusion are localized on different spatial scales. This allows us to introduce three matching zones: the diffusive zone (I) $x \ll x_A=\Gamma\rho_s$; zone (II) $\Gamma\delta=x_\delta \ll x \ll x_s=\Gamma\rho_s/\sqrt{\beta}$ of two-fluid ideal flow without the MA wave; and zone (III) $x_A \ll x \ll 1$ of two-fluid flow with the MA wave. These zones are overlapped due to the inequalities $x_\delta \ll x_A \ll x_s$ following from the assumption, $\delta \ll \rho_s \ll d_i$. The scheme of the zones is illustrated in Figs. 2(a) and 2(b). Since $\rho_s=d_i\sqrt{\beta}$, the last inequality is equivalent to $\beta \ll 1$. Internal equations (37)–(39) are simplified in these zones and the solutions are matched.

A. Diffusive zone (I) $x \ll x_A$

Since electron flow dominates in zone (I), we neglect here the v-dependent term in (37). This decouples equation (39) from (37) and (38). Solving (37) and (38), we notice that due to the $\beta\delta^2 d^2 B_z/dx^2$ term in (37) the function B_z cannot be expressed from (37) explicitly, as it is given by (46) in the beta regime (B). The transition from (B) to (C) regimes is described by two coupled second-order differential equations for B_x and B_z. The problem is still solvable analytically because in the two-fluid case the constant-B_x approximation is applicable in the narrow zone (I) at arbitrary Δ'. Indeed, the spatial scale of the eigenfunctions in zone (I) is much smaller than the electron skin depth δ. This makes the diffusive term $\delta^2 d^2 B_x/dx^2$ large in zone (I) and, correspondingly, the eigenfunction B_x is almost constant on this scale, $B_x(x)=B_x(0)=\text{const}$. In the single-fluid MHD the width of the diffusive layer at large Δ' is $\simeq\delta$, so that the constant-ψ approximation is not applicable.

According to the constant-B_x approximation, we introduce a new function $\tilde{B}_x = B_x(x)-B_x(0)$ and rewrite (38) as follows:

$$\delta^2 \frac{d^2\tilde{B}_x}{dx^2} = B_x(0) - x\frac{k^2 d_i}{\gamma\tau_a}B_z, \tag{47}$$

where the small term $\tilde{B}_x \ll \delta^2 d^2\tilde{B}_x/dx^2$ is dropped. Substituting (47) to (37) yields a nonhomogeneous parabolic cylinder equation for B_z,

$$B_z\left(\frac{\Gamma}{2\sqrt{\beta}} + \xi^2\right) - \frac{1}{4}\frac{d^2B_z}{d\xi^2} = \frac{\beta^{1/4}\sqrt{\Gamma}}{\sqrt{2}k\delta}\xi B_x(0), \tag{48}$$

where $\xi = x/(\beta^{1/4}\delta\sqrt{2\Gamma})$. The odd solution of (48) decaying at $\xi\to\infty$ is obtained by applying the Fourier transform similar to Ref. 24,

$$B_z = \frac{B_x(0)\beta^{1/4}}{U(a/2,0)}\sqrt{\frac{\Gamma}{2k^2\delta^2}}\int_0^\infty dq\; U\left(\frac{a}{2},q\right)\sin q\xi, \tag{49}$$

where $a=\Gamma/\sqrt{\beta}$. The expression for $\tilde{B}_x(x)$ in zone (I) is obtained by integrating (47) twice over x with the boundary conditions $\tilde{B}_x(0)=d\tilde{B}_x/dx(0)=0$,

$$\tilde{B}_x = -\frac{4B_x(0)\beta^{1/2}\Gamma}{U(a/2,0)}\int_0^\infty dq\frac{\sin^2(q\xi/2)}{q^2}U'\left(\frac{a}{2},q\right). \tag{50}$$

The asymptotic expansion of (50) at large $x \gg \beta^{1/4}\delta\sqrt{\Gamma}$ is determined by $U'(a/2,q)$ at $q\to 0$,

$$B_x^{(I)} = B_x(0)\left[1 - x\frac{\pi\beta^{1/4}\sqrt{2\Gamma}}{2\delta}\frac{U'(a/2,0)}{U(a/2,0)}\right]. \tag{51}$$

The ratio U'/U can be rewritten in terms of the function $G(a)$ introduced in Ref. 13,

$$\frac{U'(a/2,0)}{U(a/2,0)} = -\left(\frac{a}{2}\right)^{1/2}\frac{1}{G(a)}. \tag{52}$$

Function $G(a)$ is expressed in terms of Gamma-functions and has the following asymptotic expansions:[25]

$$G(a) = \frac{a^{1/2}\Gamma(1/4+a/4)}{2\Gamma(3/4+a/4)};$$
$$G(a) \to \frac{a^{1/2}}{2}\frac{\Gamma(1/4)}{\Gamma(3/4)}, \quad a \ll 1,$$
$$G(a) \to 1, \quad a \gg 1. \tag{53}$$

1. The case of small $\Delta' \ll \Delta'_c$

The above expressions for B_z and B_x describe universal solutions in the diffusive zone (I). At large Δ', the eigenfunctions become wider and are extended to zones (II) and (III), so that one should match B_x with the similar solution in zone (II). In a contrast to this, at small Δ', the eigenfunctions are mainly localized in zone (I). This allows us to derive the dispersion relation in this case. Indeed, substituting (52) to (50) and matching the resulting equation with (44) yields the dispersion relation for Γ at small Δ',

$$\frac{\Gamma}{G(\Gamma/\sqrt{\beta})} = \frac{\Delta'\delta}{\pi}. \tag{54}$$

Two asymptotic expressions for the growth rates are

$$\frac{\gamma\tau_a}{k} = \sqrt{\beta}\frac{d_i\Delta'\delta}{\pi}, \quad \sqrt{\beta} \ll \Delta'\delta, \tag{55}$$

$$\frac{\gamma\tau_a}{k} = d_i(\Delta'\delta)^2\left(\frac{\Gamma(1/4)}{2\pi\Gamma(3/4)}\right)^2, \quad \sqrt{\beta} \gg \Delta'\delta. \tag{56}$$

Both solutions (55) and (56) belong to the hot plasma case. Expression (55) is the case discussed in Refs. 6 and 15. It corresponds to the finite beta regime (B) when the instability is driven by the two-fluid kinetic Alfvén wave. Solution (56) corresponds to the beta regime (C) and describes the effect of saturation of the growth rate when β increases. It shows that the applicability of Refs. 6 and 15 at $\Delta'\rho_s^{1/3}\delta^{2/3} \ll 1$ is restricted by small $\beta \ll (\Delta'\delta)^2$ since the effect of B_z diffusion was ignored in these papers. The growth rate (56) coincides with the similar solutions for the whistler tearing mode.[8,10] This shows that the whistler-mediated regime of tearing instability can, in principle, be achieved within the scope of the hot plasma model at $\beta \ll 1$. Mathematically, these two limiting cases correspond to either solution (46) or (35), respectively. In both cases, the characteristic width of zone (I) is $\Delta'\delta^2$ and does not depend on β.

2. The case of large $\Delta' \gg \Delta'_c$

Considering the large Δ' case, we will solve (37)–(39) in zones (II) and (III) of ideal two-fluid plasma flow. The basic equation for $u(x)=dv/dx$ in this area is obtained by omitting diffusive terms in (37) and (38). Function $u(x)$ is proportional to $v_y(x)$ and, thus, represents the velocity component parallel to the resonant surface. Equation (37) yields the solution for B_z,

$$B_z = -\frac{\rho_s\sqrt{\beta}}{k(1+x^2/x_s^2)}\frac{du}{dx}, \tag{57}$$

while integrating (39) with the use of identity $x\,d^2B_x/dx^2 \equiv d/dx[x^2 d/dx(x^{-1}B)]$ gives

$$\frac{\gamma\tau_a}{k}u(x) = -x^2\frac{d}{dx}\frac{B_x}{x} + C. \tag{58}$$

Applying boundary conditions (44) for (58) yields the constant of integration, $C=-C_1$. The value $C_1=0$ corresponds to the case of large Δ'. Substituting (57) into (38), dividing the resulting equation by x and differentiating over x gives the second-order differential equation

$$\frac{d}{dx}\left[\left(1+\frac{x^2}{x_s^2}\right)^{-1}\frac{du}{dx}\right] = \left(1+\frac{x_A^2}{x^2}\right)\frac{u}{\rho_s^2}. \tag{59}$$

In these transformations, the term $d(B/x)/dx$ is substituted from (58). Equation (59) will be further simplified and solved in zones (II) and (III).

B. Zone (II) of two-fluid ideal flow without the MA wave ($x_\delta \ll x \ll x_s$)

In zone (II), one can omit the term x^2/x_s^2 in (59) responsible for coupling with the MA wave. This yields an equation for the ideal two-fluid flow of decoupled electrons and ions

$$\frac{d^2u}{dx^2} = \left(\frac{1}{\rho_s^2}+\frac{\Gamma^2}{x^2}\right)u. \tag{60}$$

The lhs of (60) represents the Hall term, the first term on the rhs corresponds to the $\mathbf{v}\times\mathbf{B}$ term while the second term originates from the time derivative of Faraday's law. The general solution of (60) in zone (II) is described by the superposition of two modified Bessel functions,

$$u^{(\mathrm{II})}(x) = (c_1K_{Q/2}(x/\rho_s) + c_2I_{Q/2}(x/\rho_s))\left(\frac{2x}{\pi\rho_s}\right)^{1/2}, \tag{61}$$

where $Q=\sqrt{1+4\Gamma^2}\simeq 1+2\Gamma^2$. This solution exhibits exponential decay on the ρ_s scale. This profile is a specific feature of ideal flows of decoupled electrons and ions. It corresponds formally to zero electron flow velocity in the x direction, $v_x^{(e)}=0$. Due to this, matching of the magnetic perturbations on large scales with the reconnected magnetic fields on short scales is straighforward. This is essentially a two-fluid effect, and it makes possible a broadening of the eigenfunctions up to the scales much larger than the electron skin depth δ.

For matching with zone (III), it is suitable to choose an interval $x_A \ll x \ll \rho_s$ that belongs to both zones (II) and (III). The Bessel functions are simplified in this area by making use of their asymptotic expansions at $x \ll \rho_s$,

$$u^{(\mathrm{II})}(x) = b_1x^{-\Gamma^2} + b_2(x/\rho_s)^{1+\Gamma^2}, \tag{62}$$

where $b_{1,2}$ are arbitrary constants.

First, we will match (62) with the solution in zone (I). For this purpose, we use an interval $x_\delta \ll x \ll x_A$, where zones (I) and (II) are overlapped. Inside the interval $x \ll x_A$ B_x can be found from (38) with the diffusive and xv terms dropped. This yields $B_x = xkB_z/(\Gamma\sqrt{\beta})$. Expressing B_z as a function of du/dx from (57) (without the x^2/x_s^2 term) and calculating this derivative by differentiating (62), yields

$$B_x^{(\mathrm{II})}(x) = b_1\rho_s\Gamma x^{-\Gamma^2} - b_2\frac{1+\Gamma^2}{\Gamma}x^{1+\Gamma^2} \simeq b_1\rho_s\Gamma - b_2\frac{x}{\Gamma}, \tag{63}$$

where (63) is simplified by the expansions, $x^{-\Gamma^2}\to 1$, $x^{1+\Gamma^2}\to x$ due to the smallness $\Gamma \ll 1$. We will see that at $\rho_s \gg \delta$, the growth rate Γ turns out to be small, $\Gamma \ll 1$, and, therefore, can be used as a small expansion parameter. Comparing (63) with the asymptotic expansion (51) in zone (I), yields relationships between b_1, b_2, and $B_x(0)$. Substituting them in (62) and simplifying at $\Gamma \ll 1$ similar to (63), gives an expression for $u(x)$ in zone (II),

$$u^{(\mathrm{II})}(x) = \frac{B(0)}{\rho_s\Gamma}\left(1-\frac{\pi\Gamma^3x}{2\delta G(\Gamma/\sqrt{\beta})}\right). \tag{64}$$

C. Zone (III) of ideal two-fluid flow with the MA wave $x_A \ll x \ll 1$

In this zone, one can drop the term Γ^2/x^2 in (60). Integrating the resulting equation over x yields

$$\left(1+\frac{x^2}{x_s^2}\right)(v - v^{(\infty)}) = \rho_s^2\frac{d^2v}{dx^2}, \tag{65}$$

where the constant of integration $v^{(\infty)}$ is introduced to satisfy the boundary condition (44). The solution of (65) decaying at $x\to\infty$ is described by the parabolic cylinder function,

$$v(x) - v^{(\infty)} = C_vU\left(\frac{a}{2}, \frac{x\beta^{1/4}}{\rho_s}\sqrt{\frac{2}{\Gamma}}\right), \tag{66}$$

where $a=\Gamma/\sqrt{\beta}$. The Taylor expansion of (66) at small at $x \ll \rho_s(\Gamma/\sqrt{\beta})^{1/2}$ yields an asymptotic behavior of $u(x)$ on the left boundary of zone (III),

$$u^{(\mathrm{III})}(x) = C_v\left[\frac{\beta^{1/4}}{\rho_s}\sqrt{\frac{2}{\Gamma}}U'\left(\frac{a}{2},0\right) + \frac{x}{\rho_s^2}U\left(\frac{a}{2},0\right)\right]. \tag{67}$$

Matching (67) with the corresponding asymptotics (64) in zone (II) yields a dispersion relation at large Δ' which describes the mutual effect of B_z diffusion and coupling with the MA wave,

$$\Gamma^3 = \frac{2\delta}{\pi\rho_s}G^2(\Gamma/\sqrt{\beta}). \tag{68}$$

These two effects result in a quadratic dependence on the function $G(a)<1$ in (68). Equation (68) shows that the transition into the beta regime (C) leads to the reduction of Γ in comparison with its value $\Gamma^3=2\delta/\pi\rho_s$ in the finite beta regime (B). As to the unnormalized growth rate γ, in the regime (B), Eq. (68) yields well-known "2/3" scaling on ρ_s,

$$\frac{\gamma\tau_a}{k} = (2/\pi)^{1/3}\delta^{1/3}d_i^{2/3}\beta^{1/3} \simeq \delta^{1/3}\rho_s^{2/3}, \quad m_e/m_i \ll \beta \ll \beta_c, \tag{69}$$

which predicts the increase of γ with β as $\gamma \propto \beta^{1/3}$. When beta exceeds β_c, and, correspondingly, the (C) regime begins, the ion–electron decoupling becomes less effective due to enhanced B_z diffusion and coupling with the MA wave. This leads to the saturation of γ on the level,

$$\frac{\gamma\tau_a}{k} = \sqrt{\frac{d_i\delta}{2\pi}}\frac{\Gamma(1/4)}{\Gamma(3/4)}, \quad \beta_c \ll \beta \ll 1. \tag{70}$$

Although the dependence of γ on m_i is weak, $\gamma \propto m_i^{-1/4}$, it indicates that zones (II) and (III) with intensive ion flows plays a significant role in the dynamics of the instability.

Equation (70) shows that the previous calculation of tearing mode coupling with the MA wave[13] is incomplete since the important effect of diffusion of B_z was neglected. This effect introduces an additional factor of $G(a)$. Although these two effects have different natures and are localized on the different scales, each of them add the same factor $G(a)$ in the dispersion relation (68). Matching, for example, expression (64) without B_z diffusion $[G(a) \equiv 1]$ with (67), yields the dispersion relation similar to (68), but with the first power of $G(a)$ that provides the scaling $\gamma \propto \delta^{2/5}d_i^{1/5}\rho_s^{2/5}$ calculated in Ref. 13, which is different from the correct expression (70).

A full dispersion relation that is valid at arbitrary Δ' and $\beta \ll 1$ and covers all of the two above subcases can be derived with the use of a simple heuristic approach based on the above large Δ' solution. Imposing boundary condition $v(0)=0$ on (66), gives the relationship $C_v = -\rho_s\Gamma C_2/U(a/2,0)$. Dividing Eq. (58) by x^2 and integrating from ∞ to x, yields an expression for B_x,

$$B_x(x) = C_1 + C_2x + \Gamma\rho_s x\int_x^\infty \frac{u(x')dx'}{x'^2}. \tag{71}$$

Integrating (71) by parts and dropping the integral containing du/dx, gives a good approximation for B_x at small x,

$$B_x(x) = C_1 + C_2x + C_2\frac{\Gamma^2\rho_s}{G(a)}, \tag{72}$$

where $u(x)$ and $G(a)$ are substituted from (67) and (52), and a small term of order of Γ^2 is neglected. Introducing $\Delta' = 2C_2/C_1$ and matching (72) and (51) yields the final dispersion relation,

$$\frac{\Gamma^2\rho_s}{G(\Gamma/\sqrt{\beta})} + \frac{2}{\Delta'} = \frac{2G(\Gamma/\sqrt{\beta})\delta}{\pi\Gamma}. \tag{73}$$

The effect of B_z diffusion and the coupling with the MA wave are described by the functions $G(a)$ on the rhs and the lhs of (73), respectively. The previously calculated dispersion relations[6,15] coincide with (73) at $G(a) \equiv 1$. In accordance with the definition of $G(a)$, this function satisfies a rigorous inequality $G(a) < 1$ $[G(a) \to 1$ at $a \to \infty]$. Using this property and comparing the growth rates $\Gamma^{(p)}$,[6,15] with the new ones $\Gamma^{(n)}$ given by (73), yields that rigorous inequality, $\Gamma^{(n)}(\Delta',\beta) < \Gamma^{(p)}(\Delta',\beta)$. This confirms that the B_z diffusion and the coupling with the MA wave lead to the reduction of the two-fluid effects and the growth rates. Figures 3(a) and 3(b) show that the difference between old and new results is large at large β and decreases when $\beta \to 0$. Even for small β the curves are still essentially different at small Δ' due to the difference in scalings (linear versus quadratic) on Δ'.

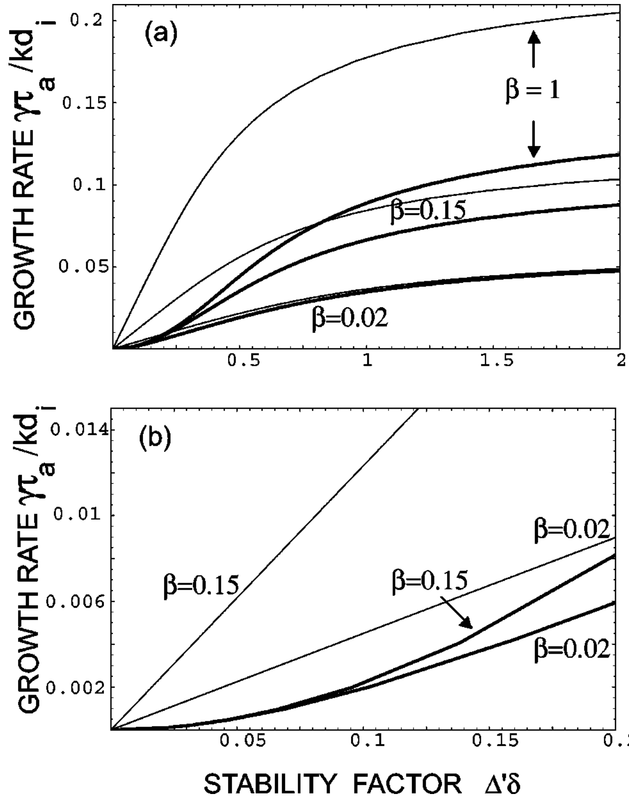

FIG. 3. Dependence of the normalized growth rate $\gamma\tau_a/d_ik$ on the stability factor Δ' at $\delta/d_i=0.02$ and three values $\beta=0.02$, 0.15, 1. (a) The thick curves are the solutions to the full dispersion relation (73) that describes the effects of the B_z diffusion and the coupling with the MA wave. The thin curves are plotted to compare these results with the previous calculations (Refs. 6 and 15) without the above two effects. (b) The importance of the B_z diffusion at small Δ' and $\beta=0.02, 0.15$ is illustrated.

VII. SUMMARY

The tearing equations (29)–(31) for the cold plasma regime and Eqs. (30), (31), and (34) for the hot plasma case follow from the general equation (19) by means of a formal expansion in cold and hot limits, respectively. In the appropriate limits, they match the equations derived previously by others. This justifies the validity of Eqs. (19) at arbitrary plasma parameters and, specifically, for the transition between cold and hot plasma regimes. This general equation predicts that the transition occurs when the time of equilibration determined by the ion-sound speed is comparable with the inverse growth rate of the instability γ^{-1}. In the transition, the intermediate situations can appear where, due to the multiscale structure of the tearing layer, the regime equivalent to the cold plasma limit exists in a wide outer area while the quasistatic regime equivalent to the hot plasma case is realized in a narrow inner area.

The important effect of resistive diffusion of the "out-of-plane" component B_z, which reduces the two-fluid effects, is included into the equations. The contribution from this effect is analyzed within the scope of the hot plasma model (kinetic Alfvén-driven instability). The corresponding growth rates are calculated at small and large Δ'. In the small Δ' case, the result is identical to the growth rate for the whistler-mediated tearing instability derived from the cold plasma model, $\beta=0$. Within the scope of the cold model, the tearing instability is driven by the combination of the compressional and shear Alfvén waves, where the CA wave is modified on short scales into whistlers. Due to the guiding magnetic field, the whistlers are strongly coupled at $k_{\parallel}=0$ to the ion motion in the compressional Alfvén wave, so that for the realization of a pure whistler regime, the ion skin depth c/ω_{pi} should be much greater than the characteristic scale of the equilibrium magnetic field (which is of the order of the plasma radius). Taking into account this condition and the typical values, $c/\omega_{pi}\simeq 5-10$ cm, one can conclude that the cold plasma model is not adequate to fast whistler-mediated reconnection in magnetically confined plasmas.

In contrast to this, the tearing instability in the hot plasma case is driven by the combination of the shear Alfvén and magneto-acoustic modes. Due to two-fluid effects, the dispersion properties of the SA wave are modified on short scales and exhibit the characteristic features of the kinetic Alfvén wave yieding the scalings $\gamma\propto\delta^{1/3}\rho_s^{2/3}$ and $\gamma\propto\delta\rho_s\Delta'$ at large and small Δ', respectively. The effect of the suppression of the growth rate by the magneto-acoustic wave was investigated at large Δ' in Ref. 13. In these calculations, the diffusion of the perturbed "out-of-plane" component of the magnetic field was ignored. We show that it is significant and changes the scalings of the growth rates in both large and small Δ' limits. The new scalings is an essential result of the paper. Specifically, the scaling at small $\Delta'\delta\ll\sqrt{\beta}\ll 1$ describes quadratic instead of linear dependence on Δ' for the collisionless tearing mode. It coincides with the growth rate of a pure electron whistler mediated tearing mode. This indicates that the regime of whistlers can be achieved at small Δ' within the scope of hot plasma model without the condition $\beta\gg 1$, which is often imposed in order to neglect the effect plasma compressibility in the induction equation. In contrast to the cold plasma model, this regime does not require large $c/\omega_{pi}\gg L$ and is realized at the rather soft condition $d_i\gg\delta$. As to the inequality $\sqrt{\beta}\gg\Delta'\delta$, it is satisfied in many cases of practical interest, for example, for the core resonant tearing modes in the reversed-field pinches. Indeed, the magnetic configuration and plasma parameters in the Madison Symmetric Torus (MST) experiments are characterized by the dimensionless factors $\Delta'\simeq 3-5$, $\delta\simeq 10^{-2}$, $\beta\simeq 0.1$, which are certainly within the scope of the above inequality. We also note that the effect of B_z diffusion on linear tearing stability was included in Ref. 12. However, that analysis was limited to resistive reconnection and treated the case of a field-reversed configuration that has no guiding field and operates at $\beta=1$. In this limit, the growth rate for the small Δ' case is reproduced by our theory despite the differences in the equilibria examined.

The importance of the above results for practical applications motivates our interest in the transition from the cold to the hot plasma cases when both the CA and the SA modes are involved in the dynamics of the instability. Strong mode localization at small Δ' simplifies analysis of their interaction in this case. The problem will be considered on the basis of the general equation (19) in a different paper.

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APPENDIX A: LINEAR EQUATIONS FOR A AND ϕ

Substituting the linearized equation (2) into (9) yields an equation for the perturbed vector and electrostatic potentials,

$$i\omega\mathbf{A}-c\,\nabla\left(\phi-\frac{p_e}{en^{(0)}}\right) =\left[\left(\frac{\mathbf{j}}{en^{(0)}}-\mathbf{v}\right)\times\mathbf{e}_y\right]B_y^{(0)}(x)+\left[\left(\frac{\mathbf{j}}{en^{(0)}}-\mathbf{v}\right)\times\mathbf{e}_z\right]B_z^{(0)} +\frac{c}{4\pi en^{(0)}}\frac{dB_y^{(0)}}{dx}\left([\mathbf{e}_z\times\mathbf{B}]+[\mathbf{e}_y\times\mathbf{B}]\frac{B_y^{(0)}(x)}{B_z^{(0)}}\right) +c\eta\mathbf{j}-\frac{m_ec}{e}\left(\frac{d\mathbf{v}_e}{dt}\right)^{(\text{linear})}, \tag{A1}$$

where the equilibrium magnetic field $\mathbf{B}^{(0)}$ and plasma current $\mathbf{j}^{(0)}$ are given by (10) and (13). The term $n^{(1)}\mathbf{j}^{(0)}/en^{(0)2}$ which results from linearization of $\mathbf{j}/en$ in (2) and takes into account the perturbation of plasma density does not contribute to (A1) because of the force-free equilibrium condition $\mathbf{j}^{(0)}\times\mathbf{B}^{(0)}=0$.

Linearizing (4) with the use of linearized continuity equation (7), $i\omega n=n^{(0)}\nabla\cdot\mathbf{v}$, and Eq. (13) gives

$$\left(\frac{d\mathbf{v}_e}{dt}\right)^{(\text{linear})}=\left[i\omega\left(\frac{\mathbf{j}}{en^{(0)}}-\mathbf{v}\right)+\frac{\nabla\cdot\mathbf{v}}{en^{(0)}}\mathbf{j}^{(0)}\right]\left(1+\frac{j_y^{(0)}k}{en^{(0)}\omega}\right) +\frac{c}{4\pi en^{(0)}}\left(\frac{j_x}{en^{(0)}}-v_x\right) \times\frac{d}{dx}\left[\left(\mathbf{e}_z+\frac{B_y^{(0)}}{B_z^{(0)}}\mathbf{e}_y\right)\frac{dB_y^{(0)}}{dx}\right], \tag{A2}$$

where the Doppler shift $j_y^{(0)}k/en^{(0)}\omega$ is caused by the small, $\propto\epsilon j_z^{(0)}/en^{(0)}$, y component of equilibrium electron flow. Since (A2) is multiplied in (A1) by a small m_e, the effect of finite electron mass results is a small correction in (A1). It is negligible everywhere except the resonant surface $x=0$, where $(d\mathbf{v}_e/dt)^{(\text{linear})}$ and the resistive term $\eta\mathbf{j}$ are the only nonzero terms on the rhs of the z component of (A1) at $x=0$,

$$i\omega A_z\left(1+\frac{j_y^{(0)}k}{en^{(0)}\omega}\right)=\left(\frac{j_x}{en^{(0)}}-v_x\right)B_y^{(0)}(x)+c\eta j_z - \frac{m_e c}{e}\left(\frac{d\mathbf{v}_e}{dt}\right)_z^{(\text{linear})}. \tag{A3}$$

These terms serve as a mean to break the frozen flux theorem. Since they are important only in the vicinity of $x=0$, one should leave in (A2) only those terms that give nonzero contribution on the rhs of (A3) at $x=0$. Since B_z is odd in x, we obtain from (23) and (B7); $\nabla\cdot\mathbf{v}|_{x=0}=v_z(0)=0$. This indicates that the term $i\omega\mathbf{j}/en^{(0)}$ on the rhs of (A2) is the most important in the tearing layer,

$$\left(\frac{d\mathbf{v}_e}{dt}\right)^{(\text{linear})}=\frac{i\omega\mathbf{j}}{en^{(0)}}. \tag{A4}$$

Substituting (A4) in (A3), using $\partial/\partial z=0$, and dropping small (proportional to $\epsilon=B_y^{(\infty)}/B_z^{(0)}\ll 1$), the Doppler terms leads to the equation for A_z,

$$-i\omega(A_z-\delta^2\nabla^2A_z)=(\alpha_{\text{H}}\nabla^2A_x+v_x)B_y^{(0)}(x), \tag{A5}$$

where the combined collisionless and resistive skin depths and the Hall factor are defined as follows: $\delta^2=c^2/\omega_{pe}^2+ic^2\eta/(4\pi\omega)$, $\alpha_{\text{H}}=c/(4\pi en^{(0)})$. Here, the plasma resistivity is assumed to be constant.

Taking the projection of (A1) on y with the gauge $\nabla\cdot\mathbf{A}=0$ gives the electrostatic potential

$$\phi-\frac{p_e}{en^{(0)}}=\frac{i}{kc}\left[\frac{\omega}{k}\frac{d}{dx}(A_x-\delta^2\nabla^2A_x)+(\alpha_{\text{H}}\nabla^2A_x+v_x)B_z^{(0)} + \alpha_{\text{H}}B_x\frac{dB_y^{(0)}}{dx}\right]. \tag{A6}$$

Substituting (A6) into the x component of (A1) yields an equation for the perpendicular component A_x that describes the perturbation of the guiding magnetic field $B_z=(i/k)\nabla^2A_x$,

$$\frac{\omega}{k}(\nabla^2A_x-\delta^2\nabla^4A_x)=ikv_zB_y^{(0)}(x)-B_z^{(0)}\nabla\cdot\mathbf{v} + \alpha_{\text{H}}\left(B_y^{(0)}(x)\nabla^2B_x-B_x\frac{d^2B_y^{(0)}}{dx^2}\right) + v_x\frac{B_y^{(0)}(x)}{B_z^{(0)}}\frac{dB_y^{(0)}}{dx}. \tag{A7}$$

Equations (A5) and (A7) are rewritten in Sec. II in terms of B_x and B_z while the last term on the rhs of (A7) is omitted. Note that Eqs. (A5) and (A7) [but not (A6)] can be alternatively derived by taking, respectively, the x and the z components of the curl of Eq. (2).

APPENDIX B: MOMENTUM AND PRESSURE EQUATIONS

Presenting small perturbations $p_{i,e}(x)\ll p_{i,e}^{(0)}$ of ion and electron pressures in the form $p_{i,e}(x,y,t)=p_{i,e}^{(0)}+p_{i,e}(x)\exp i(ky-\omega t)$ with uniform equilibrium pressures $p_{i,e}^{(0)}$, $\nabla\cdot\mathbf{j}=0$ and partial derivatives $\partial/\partial z=0$, yields linearized equations (6),

$$i\omega p_i(x)=\gamma_i p_i^{(0)}\nabla\cdot\mathbf{v}, \tag{B1}$$

$$i\omega p_e-\gamma_e p_e^{(0)}\nabla\cdot\mathbf{v}+\frac{\gamma_e}{en^{(0)}}(\mathbf{j}^{(0)}\cdot\nabla)p_e-\frac{\gamma_e p_e^{(0)}}{en^{(0)2}}(\mathbf{j}^{(0)}\cdot\nabla)n = (i\omega p_e-\gamma_e p_e^{(0)}\nabla\cdot\mathbf{v})\left(1+\frac{j_y^{(0)}k}{en^{(0)}\omega}\right)=0. \tag{B2}$$

Using $\epsilon\ll 1$ to neglect the Doppler shift in (B2) leads to the identical form of responses for ions (B1), electrons (B2) and total pressures,

$$p=\frac{c_s^2\rho^{(0)}}{i\omega}\nabla\cdot\mathbf{v}, \tag{B3}$$

where the ion sound speed is $c_s^2=(\gamma_eT_e^{(0)}+\gamma_iT_i^{(0)})/m_i$. Since in the equilibrium state, $\mathbf{v}^{(0)}=0$, the linearized equation (1) is

$$-i\omega\rho^{(0)}\mathbf{v}=-\nabla\left(p+\frac{\mathbf{B}^{(0)}\mathbf{B}}{4\pi}\right)+\frac{1}{4\pi}[(\mathbf{B}^{(0)}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{B}^{(0)}]. \tag{B4}$$

The components of (B4) are as follows:

$$-i\omega\rho^{(0)}v_x=-\frac{d}{dx}\left(p+\frac{B_z^{(0)}B_z+B_y^{(0)}B_y}{4\pi}\right)+\frac{ikB_y^{(0)}}{4\pi}B_x, \tag{B5}$$

$$-i\omega\rho^{(0)}v_y=-ik\left(p+\frac{B_z^{(0)}B_z}{4\pi}\right)+\frac{1}{4\pi}\frac{dB_y^{(0)}}{dx}B_x, \tag{B6}$$

$$-i\omega\rho^{(0)}v_z=\frac{ikB_y^{(0)}}{4\pi}B_z+\frac{1}{4\pi}\frac{dB_z^{(0)}}{dx}B_x. \tag{B7}$$

Combining (B5) and (B6) to form $\nabla\cdot\mathbf{v}$ on the lhs and substituting p in terms of $\nabla\cdot\mathbf{v}$ from (B3), yields an equation for plasma compressibility (18). Substituting perturbations of $p+B_z^{(0)}B_z/4\pi$ from (B6) to (B5) leads to the vorticity equation (17). The equation can be alternatively derived by taking the x component of the curl of Eq. (B4).

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