Relaxation to steady state in neutral-beam-injected mirrors

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The time-dependent Vlasov–Boltzmann equation is analytically studied in mirror machines with perpendicular injection. The atomic and collisional processes are taken into account, with their explicit, rigorous forms. A uniform model is assumed by considering a square-well magnetic field configuration, and the mirrors are represented by related boundary conditions on the ion distribution function. It is shown that, the ion density increases from the initial value, drops by a certain amount, then increases again up to its final value, around which it performs damped oscillations. © *1996 American Institute of Physics.* [S1070-664X(96)01104-2]

I. INTRODUCTION

The concept of a steady-state density in neutral-beam-injected mirror machines has been investigated extensively during the last 30 yrs. The earlier works concentrated on the Fokker–Planck treatment, involving nonlinear terms, and unavoidably ended up with numerical solutions.[1–4] Some analytic solutions appeared in mid 1980s in a rather crude manner, where the atomic processes were treated formally.[5,6] The more rigorous analytic treatment of the steady-state problem has been published recently,[7] where the associated atomic and collisional processes were taken into account with their explicit forms. The ion–ion angular scattering was neglected due to the fact that the ion drag on electrons is the dominant collisional process at low electron temperatures, leading to the simplification of the collisional operator, hence facilitating the analytic treatment of the problem.

The ion density in such models is generally expected to damp exponentially to a steady-state value (with no oscillations), since the charge exchange process does not affect the total number of ions, and the time evolution of the density is therefore governed by the competition between the ionization (directly proportional to the density) and the loss through the mirrors, which is proportional to density squared. However, as it was first pointed out by Ryutov, there exist kinetic effects that may lead to an instability. It is suspected that the ionization may lead the loss of the ions, since the latter is delayed by the time required for the ions to slow down from the injection velocity to the loss cone boundary. This may then deteriorate the balance between the source and loss terms, giving rise to an instability. A kinetic description of plasma dynamics is therefore necessary to study the stability of the steady state, subject to small-density perturbations, varying in time with the characteristic scale of the order of ion lifetime, in order to obtain a complete prediction of the ion behavior.

The present goals designated for the mirror system are somewhat diverted from being an ultimate fusion reactor to mirror-based neutron generators, for material testing of the first wall of tokamak reactors[8] and other possible applications. This investigation is expected to contribute to the understanding of the dynamics of fast tritium or deuterium components in different schemes of such generators as well. It is also of interest from the general point of view, for instance, the macroscopic description (in terms of density) of systems, strongly in nonequilibrium within a time scale comparable with the intrinsic time of particle interactions.

First, rough approaches to this problem are presented in Refs. 5 and 6, where stability analysis has been carried out with constant and zero charge exchange rates, respectively. In the present work, the problem is analyzed rigorously by writing the time-dependent Vlasov–Boltzmann equation, with the explicit expressions for the relevant collisional and atomic processes. The model described in Ref. 7 is adopted and the time-dependent solution for the ion distribution function is analytically obtained. Using this solution, the time evolution of the ion density is illustrated from the moment the neutral beam is injected. It is seen to evolve exponentially during two consecutive time intervals in a sawtooth manner, after which a relaxation to steady state through damped oscillations prevails.

II. THEORY

The complete form of the Vlasov–Boltzmann equation for the ion distribution function can be formally written as

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{F}}{M}\cdot\frac{\partial f}{\partial \mathbf{v}}=-\nabla_v\cdot\mathbf{j}+Q(\mathbf{v}), \tag{1}$$

where f and M denote the ion distribution function and the ion mass, respectively, $\mathbf{j}$ is the ion flux in velocity space due to collisions, and $Q(\mathbf{v})$ represents the source and loss terms for ions with velocity $\mathbf{v}$. In neutral-beam-injected mirrors, the source terms consist of the ionization of the beam by the electron impact and charge exchange with the ions. The loss terms, on the other hand, consist of the neutralization of ions due to charge exchange with the beam and the mirror losses. As mentioned in the Introduction, the loss rate through the mirrors will not be considered explicity in the term $Q(\mathbf{v})$, but will be taken into account as a loss cone boundary, on the surface of which the ion distribution drops to zero. Adopting the expressions derived for $\mathbf{j}$ and $Q(\mathbf{v})$ in Ref. 7 and letting the magnetic field be in the z direction, Eq. (1) for the uniform model can be written as

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$$\frac{\partial f}{\partial t}-\omega_c\,\frac{\partial f}{\partial\varphi}=C\boldsymbol{\nabla}_v\cdot(\mathbf{v}f\,)+n_b\delta(\mathbf{v}-\mathbf{v}_b)\bigg(n\langle\sigma_i v_e\rangle+\int\sigma_{\rm ex}(|\mathbf{v}-\mathbf{v}_b|)|\mathbf{v}-\mathbf{v}_b|f\;d^3v\bigg)-n_b\sigma_{\rm ex}(|\mathbf{v}-\mathbf{v}_b|)|\mathbf{v}-\mathbf{v}_b|f, \tag{2}$$

where $\omega_c=eB/Mc$, e is the ion charge, B is the magnetic field, φ is the azimuthal angle, C is the same term defined in Ref. 7, n_b and n are the beam and plasma densities, respectively, $\mathbf{v}_b$ is the beam velocity, v_e is the electron speed, and finally σ_i and $\sigma_{\rm ex}$ are the electron impact ionization and charge exchange cross sections, respectively. Due to the fact that ω_c is usually much larger than the other frequencies involved in Eq. (2), $\partial f/\partial\varphi$ must be very small. Since the variations with φ are periodic, this can be possible only if f consists of a large, φ independent part (f_0) and a small φ-dependent part (f_1), that is,

$$f(\mathbf{v},t)=f_0(\mathbf{v},t)+f_1(\mathbf{v},t),$$

where $f_0(\mathbf{v},t)\gg f_1(\mathbf{v},t)$. Using this expansion, the first-order form of Eq. (2) yields

$$\frac{\partial f_0}{\partial t}-\omega_c\,\frac{\partial f_1}{\partial\varphi}=C\boldsymbol{\nabla}_v\cdot(\mathbf{v}f_0)+n_b\delta(\mathbf{v}-\mathbf{v}_b)\bigg(n\langle\sigma_i v_e\rangle+\int\sigma_{\rm ex}(|\mathbf{v}-\mathbf{v}_b|)|\mathbf{v}-\mathbf{v}_b|f_0\;d^3v\bigg)-n_b\sigma_{\rm ex}(|\mathbf{v}-\mathbf{v}_b|)|\mathbf{v}-\mathbf{v}_b|f_0. \tag{3}$$

Adopting the spherical coordinates in velocity space, with φ remaining as the azimuthal angle, setting the beam velocity arbitrarily in the x direction, and taking the average value of the resulting form of Eq. (3) with respect to φ, one obtains

$$\frac{\partial\psi(v,\theta,t)}{\partial t}=C'vn(t)\,\frac{\partial\psi(v,\theta,t)}{\partial v}-\frac{n_b g(v,\theta)}{2\pi}\,\psi(v,\theta,t)+\frac{v^3}{v_b^2}\,S(t)\,\delta(v-v_b)\,\delta\bigg(\theta-\frac{\pi}{2}\bigg), \tag{4}$$

where $\psi(v,\theta,t)=v^3f_0(v,\theta,t)$, $C'=C/n(t)$,

$$g(v,\theta)=\int_0^{2\pi}\sigma_{\rm ex}(|\mathbf{v}-\mathbf{v}_b|)|\mathbf{v}-\mathbf{v}_b|d\varphi, \tag{5}$$

and

$$S(t)=\frac{n_b}{2\pi}\bigg(n(t)\langle\sigma_i v_e\rangle+\int\;\frac{dv}{v}\int_0^{\pi}\psi(v,\theta,t)\times g(v,\theta)\sin\,\theta\;d\theta\bigg). \tag{6}$$

The dependence of each quantity involved in Eq. (4) on the independent variables is illustrated explicity, to serve as a reference for the proceeding analysis.

At this moment, two of the independent variables, v and t, will be transformed to a set of new variables, $\tau(t)$ and $u(v,t)$, via the relations

$$\tau(t)=\int_0^t C'n(t')dt' \tag{7}$$

and

$$u(v,t)=ve^{\tau(t)}. \tag{8}$$

The corresponding form of Eq. (4) can be readily expressed as

$$C'n(\tau)\,\frac{\partial\psi(u,\theta,\tau)}{\partial\tau}+\frac{n_b g(ue^{-\tau},\theta)}{2\pi}\,\psi(u,\theta,\tau)=\frac{u^3e^{-3\tau}}{v_b^2}\,S(\tau)\,\delta(ue^{-\tau}-v_b)\,\delta\bigg(\theta-\frac{\pi}{2}\bigg). \tag{9}$$

The solution of the homogeneous part is

$$\psi_h(u,\theta,\tau)=\psi_h(u,\theta,0)\exp\bigg(-\frac{n_b}{2\pi C'}\times\int_0^{\tau}\frac{g(ue^{-\tau'})}{n(\tau')}\,d\tau'\bigg).$$

Applying a standard technique, the variation of parameters, by replacing, $\psi_h(u,\theta,0)$ with $K(u,\theta,\tau)$, yields the following expression for the complete solution of Eq. (9):

$$\psi(u,\theta,\tau)=\bigg\{K(u,\theta,0)+\frac{u^3\delta(\theta-\pi/2)}{C'v_b^3}\times\int_0^{\tau}\frac{e^{-3\tau'}}{n(\tau')}\,S(\tau')\,\delta\bigg[\tau'-\ln\bigg(\frac{u}{v_b}\bigg)\bigg]\times\exp\bigg(\frac{n_b}{2\pi C'}\int_0^{\tau}\frac{g(ue^{-\tau''})}{n(\tau'')}\,d\tau''\bigg)d\tau'\bigg\}G(u,\tau), \tag{10}$$

where

$$G(u,\tau)=\exp\bigg(-\frac{n_b}{2\pi C'}\int_0^{\tau}\frac{g(ue^{-\tau'})}{n(\tau')}\,d\tau'\bigg).$$

The term in the curly brackets represents the envelope of the function $G(u,\tau)$. The instant $t=0$ (injection of the beam) corresponds to $\tau=0$, and it can be seen that, at this instant, the second term in curly brackets is zero. The term, $K(u,\theta,0)$ should therefore represent the ions, initially present in the system. This term, and hence the function $\psi(u,\theta,\tau)$ described by Eq. (10), can be related to the initial distribution function $f_0(v,\theta,0)$, by noting that

$$K(u,\theta,0)=\psi(u,\theta,0)=u^3f_0(u,\theta,0),$$

where the fact that $u=v$ at $t=0$ is used.

Although $K(u,\theta,0)$ is τ independent, the dependence on u leads to a dependence on time t, that is

$$K(v,\theta,t)=v^3e^{3\tau(t)}f_0(ve^{\tau(t)},\theta,0).$$

This behavior implies a shift toward $v\sim0$ with time. For instance, if the initial distribution is a localized function around a velocity v^*, given by

$$f_0(v,\theta,0)\,\alpha\,\delta(v-v^*)\,\delta(\theta-\pi/2),$$

then

$$K(v,\theta,t)\alpha v^3 e^{2\tau(t)}\delta(v-v^* e^{-\tau(t)})\delta(\theta-\pi/2),$$

which illustrates that the initially existing ion distribution shifts toward $v\sim 0$ at later times, and should leave the system at a certain instant.

After having devoted the necessary attention to the meaning of the term $K(u,\theta,0)$ in Eq. (10), we may now evaluate the integral, involving the delta function, to obtain

$$\psi(u,\theta,\tau)=\left[K(u,\theta,0)+\frac{\delta(\theta-\pi/2)S(\ln u/v_b)}{C'n(\ln u/v_b)}\times\exp\left(\frac{n_b}{2\pi C'}\int_0^{\ln(u/v_b)}\frac{g(ue^{-\tau'})}{n(\tau')}\,d\tau'\right)\right]G(u,\tau),$$

for $0<\ln(u/v_b)<\tau$, and $\psi(u,\theta,\tau)=K(u,\theta,0)$; $G(u,\tau)$ otherwise. Noting that $\ln(u/v_b)=\ln(ve^{\tau}/v_b)=\tau-\ln(v_b/v)$, the expression above can be rewritten as

$$\psi(v,\theta,t)=K(v,\theta,t)G[ve^{\tau(t)},\tau(t)]+\frac{\delta(\theta-\pi/2)S[\tau(t)-\xi]}{C'n[\tau(t)-\xi]}\times\exp\left(-\frac{n_b}{2\pi C'}\int_{\tau(t)-\xi}^{\tau(t)}\frac{g(ve^{\tau(t)-\tau'})}{n(\tau')}\,d\tau'\right),\tag{11}$$

where $\xi=\ln(v_b/v)$. The expression above is valid for the interval $v_b e^{-\tau}<v<v_b$; otherwise only the first term on the right-hand side survives. Equation (11) is actually an integral equation for $\psi(v,\theta,t)$, since it exists in the function S and the density terms on the right-hand side, as indicated by Eq. (6) and the expression

$$n(t)=2\pi\int f_0(v,\theta,t)v^2\sin\theta\,d\theta=2\pi\int_{v_0}^{\infty}\int_0^{\pi}\frac{\psi(v,\theta,t)}{v}\sin\theta\,d\theta\,dv,\tag{12}$$

where v_0 denotes the loss cone boundary.[7] Equations (6), (11), and (12) therefore constitute a set of coupled integral equations, to be solved simultaneously, for obtaining the time evolution of the ion distribution function and density.

The difficulty to be encountered during this procedure can be somewhat reduced by adopting Eq. (4) instead of Eq. (6). Multiplying each term in Eq. (4) by $2\pi\sin\theta/v$, integrating over θ and v, and using Eq. (12), yields

$$\frac{\partial n}{\partial t}=-2\pi C'n\int_0^{\pi}\psi(v_0,\theta,t)\sin\theta\,d\theta+n_b n\langle\sigma_i v_e\rangle.\tag{13}$$

It can easily be seen that, by substituting Eq. (11) into Eq. (13) yields an equation, relating the function S to the ion density and its derivative with respect to time. Substituting Eq. (11) into Eq. (12) and using the relation described above yield an equation, involving only the ion density and the system parameters. Before proceeding in this prescribed direction, further simplifications can be introduced by analyzing the evolution of the ion density piecewise in time. According to the earlier discussions, it requires a certain time for the initial distribution, represented by $K(u,\theta,0)$, to shift to the velocity v_0, and a longer time for the distribution of the beam particles to broaden down to the same velocity. Therefore, the function $\psi(v_0,\theta,t)$ in Eq. (13) should remain zero until a certain time t_1, required for the term $K(u,\theta,0)$, to shift down to the velocity v_0, and the ion density in the time interval $0<t<t_1$ is given by

$$n(t)=n(0)e^{n_b\langle\sigma_i v_e\rangle t},\tag{14}$$

where $n(0)$ is the density at $t=0$. At the instant t_1, there will be a sudden drop in the ion density, since the particles represented by $K(u,\theta,0)$ leave the system. The velocities of the beam particles remain to be larger than v_0, and hence the ion density continues to obey the time dependence given in Eq. (14) for $t>t_1$, but has a different starting value, defined by the sudden drop. This behavior ends at a certain instant t_2 ($t_2>t_1$), representing the time required for the beam distribution to broaden down to the velocity v_0, since the function $\psi(v_0,\theta,t)$ in Eq. (13) becomes effective for the time range $t>t_2$. After having described the first two phases of the density evolution, we shall now concentrate on the remaining phase $t>t_2$, to investigate the stability.

According to the prescribed procedure, Eq. (11) will be substituted into Eq. (13) first. However, it should be noted that for the range $t>t_2$, the first term on the right-hand side does not exist anymore, since the original ions have already left the system. This is the major simplification, provided by the piecewise analysis in time. The substitution mentioned above yields

$$\frac{\partial n}{\partial t}=-2\pi n\frac{S[\tau(t)-\tau_0]}{n[\tau(t)-\tau_0]}\exp\left(-\frac{n_b}{2\pi C'}\times\int_{\tau(t)-\tau_0}^{\tau(t)}\frac{g(v_0e^{\tau(t)-\tau'})}{n(\tau')}\,d\tau'\right)+n_b n\langle\sigma_i v_e\rangle,$$

where $\tau_0=\ln(v_b/v_0)$. Since the right-hand side of this expression is essentially in terms of τ, it is convenient to write the left-hand side in terms of the same variable as well. This manipulation gives

$$\frac{\partial n(\tau)}{\partial\tau}=\frac{n_b\langle\sigma_i v_e\rangle}{C'}-\frac{2\pi}{C'}\frac{S(\tau-\tau_0)}{n(\tau-\tau_0)}\cdot\exp\left(-\frac{n_b}{2\pi C'}\times\int_{\tau-\tau_0}^{\tau}\frac{g(v_0e^{\tau-\tau'})}{n(\tau')}\,d\tau'\right).\tag{15}$$

It should be noted that $\tau>\tau_0$ for the corresponding range $t>t_2$. Now, Eq. (11) is substituted into Eq. (12), with the first term on the right-hand side being zero again, and the resulting expression can be written in terms of the variable τ as

$$n(\tau)=\frac{2\pi}{C'}\int_0^{\tau_0}\frac{S(\tau-\xi)}{n(\tau-\xi)}\exp\left(-\frac{n_b}{2\pi C'}\times\int_{\tau-\xi}^{\tau}\frac{g(v_b e^{\tau-\xi-\tau'})}{n(\tau')}\,d\tau'\right)d\xi.\tag{16}$$

Replacing τ by $\tau+\tau_0-\xi$ in Eq. (15) yields

$$\frac{\partial n(\tau+\tau_0-\xi)}{\partial\tau}=\frac{n_b\langle\sigma_i v_e\rangle}{C'}-\frac{2\pi}{C'}\frac{S(\tau-\xi)}{n(\tau-\xi)}\exp\left(-\frac{n_b}{2\pi C'}\times\int_{\tau-\xi}^{\tau+\tau_0-\xi}\frac{g(v_b e^{\tau-\xi-\tau'})}{n(\tau')}\,d\tau'\right). \tag{17}$$

Evaluating the ratio $S(\tau-\xi)/n(\tau-\xi)$ from Eq. (17) and substituting into Eq. (16), one obtains

$$n(\tau)=\int_0^{\tau_0}\left(-\frac{\partial n(\tau+\tau_0-\xi)}{\partial\tau}+\frac{n_b\langle\sigma_i v_e\rangle}{C'}\right)\exp\left(\frac{n_b}{2\pi C'}\times\int_{\tau}^{\tau+\tau_0-\xi}\frac{g(v_b e^{\tau-\xi-\tau'})}{n(\tau')}\,d\tau'\right)d\xi. \tag{18}$$

The above equation describes the evolution of the ion density for $t>t_2$ or $\tau>\tau_0$, that is, after the initially existing ions have left the system, and the beam distribution has broadened down to the loss cone boundary. In this sense, it describes the long time behavior of the system, and can be used for the linear analysis of the stability. Assuming that a steady-state value of the density n_0 exists, we shall iterate n in Eq. (18) around this value, by letting

$$n(\tau)=n_0+n_1(\tau),\quad (n_1\ll n_0).$$

Substituting into Eq. (18), the zeroth, and first-order equations can be written as

$$n_0=\frac{n_b\langle\sigma_i v_e\rangle}{C'}\int_0^{\tau_0}\exp\left(\frac{n_b}{2\pi C' n_0}\times\int_{\tau}^{\tau+\tau_0-\xi}g(v_b e^{\tau-\xi-\tau'})d\tau'\right)d\xi, \tag{19}$$

$$n_1(\tau)=-\int_0^{\tau_0}\left(\frac{\partial n_1(\tau+\tau_0-\xi)}{\partial\tau}+\frac{n_b^2\langle\sigma_i v_e\rangle}{2\pi C'^2 n_0^2}\times\int_{\tau}^{\tau+\tau_0-\xi}g(v_b e^{\tau-\xi-\tau'})n_1(\tau')d\tau'\right)\times\exp\left(\frac{n_b}{2\pi C' n_0}\int_{\tau}^{\tau+\tau_0-\xi}g(v_b e^{\tau-\xi-\tau'})d\tau'\right)d\xi. \tag{20}$$

At this moment it should be noted that, by changing variables in Eq. (19) the τ dependence dissappears, and the resulting expression corresponds exactly to the integration of the steady-state ion distribution function, obtained in Ref. 7.

Letting $\xi=\tau_0-x$ and $\tau'=\tau+y$, Eq. (20) can be rewritten as

$$n_1(\tau)=-\int_0^{\tau_0}h(x)\frac{\partial n_1(\tau+x)}{\partial\tau}\,dx-\frac{n_b^2\langle\sigma_i v_e\rangle}{2\pi C'^2 n_0^2}\times\int_0^{\tau_0}\int_0^{x}g(v_0 e^{x-y})n_1(\tau+y)h(x)dy\,dx, \tag{21}$$

where

$$h(x)=\exp\left(\frac{n_b}{2\pi C' n_0}\int_0^{x}g(v_0 e^{x-y})dy\right). \tag{22}$$

It can be easily illustrated that the double integral can be rearranged, so that Eq. (21) takes the following form;

$$n_1(\tau)=-\int_0^{\tau_0}h(x)\frac{\partial n_1(\tau+x)}{\partial\tau}\,dx-\frac{n_b^2\langle\sigma_i v_e\rangle}{2\pi C'^2 n_0^2}\int_0^{\tau_0}dy\;n_1(\tau+y)\times\int_y^{\tau_0}g(v_0 e^{x-y})h(x)d(x). \tag{23}$$

Since $\partial n_1(\tau+x)/\partial\tau$ can be written as $\partial n_1(\tau+x)/\partial x$, the first term on the right-hand side can be integrated by parts to yield

$$n_1(\tau+\tau_0)=\frac{n_b}{2\pi C' n_0 h(\tau_0)}\left(\int_0^{\tau_0}n_1(\tau+x)h(x)g(v_0 e^{x})dx-\frac{n_b\langle\sigma_i v_e\rangle}{C' n_0}\int_0^{\tau_0}dy\;n_1(\tau+y)\times\int_y^{\tau_0}g(v_0 e^{x-y})h(x)d(x)\right).$$

Interchanging x with y in the first term on the right-hand side and combining the two integrals, one finally obtains

$$n_1(\tau+\tau_0)=\int_0^{\tau_0}n_1(\tau+y)H(y)dy, \tag{24}$$

where

$$H(y)=\frac{n_b}{2\pi C' n_0 h(\tau_0)}\left(h(y)g(v_0 e^{y})-\frac{n_b\langle\sigma_i v_e\rangle}{C' n_0}\int_y^{\tau_0}g(v_0 e^{x-y})h(x)d(x)\right), \tag{25}$$

which is a known function, determined by the system parameters.

To obtain the time evolution of the perturbation density n_1, the Fourier representation will be used;

$$n_1(\tau)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}n_\omega e^{-i\omega\tau}\,d\omega.$$

Substitution into Eq. (24) yields the dispersion relation,

$$e^{-i\omega\tau_0}=\int_0^{\tau_0}H(y)e^{-i\omega y}\,dy. \tag{26}$$

It is prohibitively difficult to solve for ω analytically, with the existing form of the function $H(y)$. At this stage we shall resort to the typical ranges of the parameters involved. For such ranges, it was shown[7] that the function g remains almost constant, and consequently

$$n_0=\frac{g n_b\tau_0}{2\pi C'\ \ln[1+(g/2\pi\langle\sigma_i v_e\rangle)]}.$$

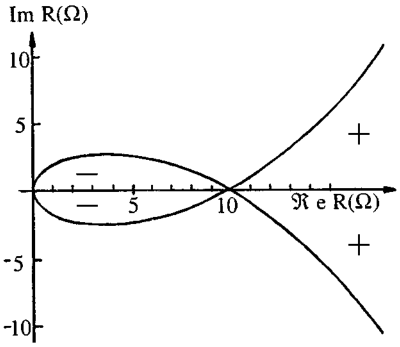

FIG. 1. The right-hand side of Eq. (28), $R(\Omega)$, plotted on the complex plane for real Ω. The signs of the areas indicate the sign of the imaginary part of Ω, which would map $R(\Omega)$ into that particular region.

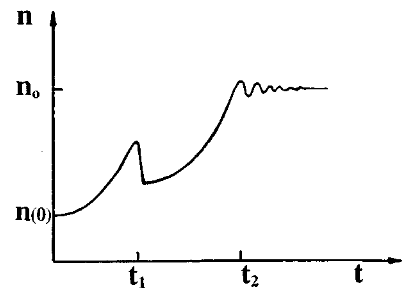

FIG. 2. The qualitative illustration of the time evolution of the ion density, with $t=0$ corresponding to the instant the neutral beam is injected.

Using the fact that $g\sim$const and the expression above in Eqs. (22) and (25) yields

$$H(y)=\frac{2\pi\langle\sigma_i v_e\rangle}{g\tau_0}\ln\left(1+\frac{g}{2\pi\langle\sigma_i v_e\rangle}\right)\times\left\{\exp\left[\frac{y}{\tau_0}\ln\left(1+\frac{g}{2\pi\langle\sigma_i v_e\rangle}\right)\right]-1\right\}. \quad (27)$$

Substituting into Eq. (26), one obtains

$$\frac{2}{\lambda}=\Omega^2+i\,\frac{\Omega^2}{\tan(\Omega/2)}, \quad (28)$$

where $\Omega=\omega\tau_0$ and

$$\lambda=\frac{g}{2\pi\langle\sigma_i v_e\rangle\ln^2[1+(g/2\pi\langle\sigma_i v_e\rangle)]}.$$

The stability condition can now be investigated by mapping the complex Ω plane onto the right-hand side of Eq. (28), denoted as $R(\Omega)$. The plot of the function $R(\Omega)$ on the complex plane for real Ω is illustrated in Fig. 1. The signs of the areas indicate the sign of the imaginary part of Ω, which would map $R(\Omega)$ into that particular region. The left-hand side of Eq. (28), on the other hand, is purely real and can easily be shown to lie within the interval $0<(2/\lambda)\leqslant 1.3$ for any set of nonzero values of the parameters concerned. This clearly implies that Eq. (28) can be satisfied only if ω has a negative imaginary part and a nonzero real part, hence the ion density exhibits damped oscillations around the steady-state value n_0, for $t>t_2$. The time evolution of the ion density, from the moment the beam is injected until the steady state, is now completely described and illustrated qualitatively in Fig. 2.

III. CONCLUSION

In this work, we have studied the time evolution of the ion density in mirror machines with perpendicular injection, to investigate the stability of the steady state, subject to small density perturbations. The time-dependent Vlasov–Boltzmann equation is written with the explicit expressions for the collisional, ionization, and charge exchange processes, and solved analytically for the typical ranges of parameters involved. To the best of our knowledge, this is the only analytical investigation of the problem, with the charge exchange process being taken into account realistically.

The results obtained imply the existence of three different phases in the time evolution of the ion density, from the moment the beam is injected until the steady state. In the first phase, the density grows exponentially due to the accumulation of ions supplied by the beam on the initially existing ions, which gradually shift toward the loss cone. The second phase starts with a drop in the density, at the time the initially existing ions leave the system through the loss cone. The abruptness of the drop depends on the steepness of the initial distribution. This phase continues with the same exponential growth, produced by the ionization of the beam, until the velocity distribution of these ions broadens down to the loss cone. Then, the last phase starts, where the loss mechanism begins to compensate the accumulation. Using the Fourier representation and mapping the complex frequency into the consequent "dispersion relation," it is shown that the ion density finally relaxes to a steady state value through damped oscillations, confirming the stability of the steady state. It may therefore be concluded that the kinetic effects pointed out in the Introduction, as possible reasons to expect the development of an instability, proved to be effective in converting the exponential-like relaxation to an oscillatory one, but quantitatively inadequate for causing an instability.

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