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Exact relativistic expressions for polarization of incoherent Thomson scattering

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We present a derivation of the degree of polarization for incoherent Thomson scattering (TS) using Mueller matrix formalism. An exact analytic solution is obtained for spectrum-integrated matrix elements. The solution is valid for the full range of incident polarizations, scattering angles, and electron thermal motion from non-relativistic to ultra-relativistic. It is based on a newly developed theoretical model, a finite transit time (FTT) correction to previous theoretical work on TS polarization. The Mueller matrix elements are substantially different from previous calculations without the FTT correction, even to the lowest linear order in $T_e/m_ec^2 \ll 1$. Mathematically, the derivation is a unique example of fully analytical integration of the 3D scattering operator over a relativistic Maxwellian distribution function; experimentally, the results have application to the use of the polarization properties of Thomson scattered light as a method of electron temperature measurement. The results can also be used as a reliable tool for benchmarking and verification of numerical codes for frequency resolved properties of TS polarization. *Published by AIP Publishing.*
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I. INTRODUCTION

Incoherent Thomson scattering (TS) is routinely used for electron temperature measurement, with T_e proportional to the square of the spectral width of the scattered light.[1] Instead of the frequency spectrum broadening, we analyze here the polarization properties of the TS radiation as a method of electron temperature measurement.

The term "depolarization" has been widely used in the TS literature for many years. Indeed, the scattering process changes the polarization of the light, an effect that becomes significant in high-temperature plasmas and is typically described by the relativistic depolarization factor q (see Refs. 1–4). This factor quantifies the reduction of scattered spectral intensity when the scattered light collection system selects for a specific orientation of linear polarization and is due to relativistic terms $\propto v_{Te}^2/c^2$ in the polarization part of the scattering operator.

Although this reduction is referred to as depolarization, this usage differs from the use of depolarization in the present paper. Indeed, the aforementioned reduction of intensity takes place even for scattering on a single moving electron. In this case, the scattered electromagnetic wave has a Doppler-shifted frequency but still remains monochromatic and completely polarized. The transition from fully polarized incident light to partially polarized TS radiation is caused by the superposition effect of a large number of randomly moving electrons. It results in broadening of the frequency spectrum and also renders the scattered radiation partially polarized even though the incident light is fully polarized. We focus our attention on this mechanism of loss of polarization in the process of incoherent TS.

The loss of polarization is quantified by the *degree of polarization P*, or equivalently by the *degree of depolarization* $D = 1 - P$. The possibility of determining the plasma electron temperature by measuring the degree of depolarization was suggested as early as 1968 in Ref. 5 and more recently in Ref. 6. If the degree of polarization dependence on electron temperature is accurately known from theory, the accuracy of such a diagnostic could potentially exceed that of the conventional spectrum-based TS method. First order in T_e/m_ec^2 effects were analyzed theoretically in Refs. 5, 7, and 8. Thus motivated, we revisited this problem to analyze whether polarization effects may be suitable for application to advanced TS diagnostics on ITER.

The most complete description of the polarization is based on the Mueller matrix formalism. The 4×4 Mueller matrix links the Stokes vectors of the incident and scattered light. Our previous publications[9,10] were mostly devoted to parametric studies of the degree of polarization. Possible implementations of a polarization-based T_e diagnostic were discussed in Ref. 11. In all these papers, expressions for the Mueller matrix elements were presented without derivation. The purpose of the present paper is, first, to develop a self-consistent theoretical model of the TS polarization and, second, to describe the derivation of the exact relativistic expressions for the Mueller matrix elements. The first calculation of the Mueller matrix for Thomson scattering was performed in Ref. 8 in 2000 and limited to first order terms in $T_e/m_ec^2 \ll 1$. In contrast to Ref. 8, we present here an advanced Mueller matrix theory that results in matrix elements obtained from exact analytical calculations. They are expressed in a compact form after analytical integration of a three-dimensional, relativistic scattering operator over a relativistic Maxwellian distribution function and universally valid for the full range of electron thermal motion from non-relativistic to ultra-relativistic. Low temperature expansions at $T_e/m_ec^2 \ll 1$ are not in agreement with Ref. 8 due to the incorrect form of the scattering operator chosen as the starting point of the model.[8] The existence of the analytical solution is of a principal importance for optimization of the polarization-based TS diagnostic setup over multidimensional parameter space of the problem.[10,11]

Following the general approach accepted for incoherent TS calculations, we consider, first, polarization properties for scattering on a single electron. They are described by Lienard-Wiechert solution for the scattered electric field reemitted by an electron moving along the unperturbed trajectory with arbitrary relativistic velocity and oscillating in the field of the incident monochromatic wave. The starting point of the Mueller matrix formalism is a transformation of the time averaging in the definition of the Stokes vector components to integration over the spectrum. The resulting expressions are linked with the frequency-integrated products of the Fourier components of the truncated fields in Appendix A 2. They allow us to define in Sec. II a set of three Mueller matrices and the corresponding Stokes vectors (auxiliary, spectral, and frequency integrated) describing the change of polarization in the process of scattering on an individual electron moving in unbounded space.

The zero-component of the spectral Stokes vector corresponds to the spectral intensity from a single electron. Our result for this component is consistent with the expression derived in the first part of Ref. 12 devoted to the infinite scattering volume (infinite transit time, ITT) case. It yields the spectral intensity on the detector $P^{(single)} \propto \delta(\omega - \omega_d)/(1-\beta_s)^6$ scattered by a single electron moving with the velocity $\boldsymbol{\beta} = \mathbf{v}/c$ in the infinite scattering volume, where β_s is the projection of $\boldsymbol{\beta}$ on the scattered wave direction and ω_d is the Doppler shifted frequency of the wave. The key dependences of $P^{(single)}$ on ω and β_s are derived in Sec. II B and Appendix A 3. They are also identical to Equation (7.2.19) in Ref. 2 and Equation (4.35) in Ref. 14. According to the terminology of Ref. 2, the spectral intensity $P^{(single)}$ represents the *time-at-observer* power from a single electron.

The result of Ref. 12 was declared in Ref. 13 (1980) to be incorrect due to improper handling of the square of a δ-function in the spectral power equation. The arguments of Ref. 13 have been reviewed recently in a detailed tutorial article.[14] Performing our calculations, we found that the square of a δ-function was properly treated in mathematical transformations in the first ITT part of Ref. 12. The inconsistency between Ref. 13 and Ref. 12 originates not from a mathematical mistake in Ref. 12 but from the erroneous comparison of the *time-at-particle* power treated in Ref. 13 with the *time-at-observer* power analyzed in Refs. 2, 12, and 14.

Using the single electron Mueller matrices obtained in Sec. II allows us to account for the effect of many particles in Sec. III. The combined effect of many electrons was originally expressed in Ref. 12 by two different weighting factors used for averaging over Maxwellian distribution function: by the weighting factor $P_{ITT} \propto (1-\beta_s)^{-6}$ presented in the first ITT part of Ref. 12 and another finite transit time (FTT) scaling, $P_{FTT} \propto (1-\beta_s)^{-5}$, derived in the second part of Ref. 12 devoted to the finite scattering volume (finite transit time, FTT) regime. The ITT scaling corresponds to the instantaneous spectral intensity $P^{(single)}$ from a single electron multiplied by the number of particles stationary residing in the scattering volume. This operation is invalid in the case of finite scattering volume due to the interruption of radiation caused by the boundaries of the scattering zone. The FTT scaling has an additional factor $(1-\beta_s)$ compared to the ITT expression. This factor takes into account modification of the mean power on the detector due to the impulsive character of the scattered radiation. The FTT form of the weighting factor is generally accepted in all present-day relativistic treatments of Thomson scattered radiation. We use this expression for our polarization calculations contrary to Ref. 8, where the incorrect ITT weighting factor was used for averaging over Maxwellian distribution function.

The second important improvement is the optimal choice of reference frame for averaging over velocity space. This allows us to perform analytical integration of the Mueller matrix elements over relativistic Maxwellian distribution function in Sec. IV. The derivation yields an exact relativistic expression for the degree of depolarization which spans the full range of incident polarizations, scattering angles, and electron thermal motion from non-relativistic to ultra-relativistic. Both of these improvements differ significantly from Ref. 8, where only an approximate solution, limited by linear in T_e/m_ec^2 corrections to the cold plasma case, was calculated on the basis of the incorrect ITT weighting factor. Our technique of exact integration can also be formally applied to the ITT weighting factor used in Ref. 8. This also yields Mueller matrix elements valid at all temperatures. Their low temperature limits verify the first-order expansions in T_e obtained in Ref. 8 for the ITT model and increases confidence in both the first-order calculations[8] and the correctness of our scheme of exact analytical integration. The corresponding mathematical transformations and some comments about ITT and FTT effects are presented in Appendix B.

II. THOMSON SCATTERING FROM A SINGLE ELECTRON

The polarization properties of a non-monochromatic plane wave are characterized by the complex coherency matrix $\mathbf{J}$. The matrix is constructed from time averaged quadratic combinations of the field components (see Ref. 15). It is represented, in general, by four real quantities which can be equivalently expressed by four Stokes parameters or the 4-component Stokes vector $\mathbf{S}$

$$\mathbf{J} = \begin{pmatrix} \overline{E_xE_x^\star} & \overline{E_xE_y^\star} \\ \overline{E_yE_x^\star} & \overline{E_yE_y^\star} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{pmatrix}. \quad (1)$$

The S_0 component is a measure of the total intensity I of the wave, and the remaining components describe the polarization properties. For a purely monochromatic, fully polarized incident wave, the amplitudes and the phases of E_x and E_y are independent of time. In this case det $|\mathbf{J}| = 0$, leading to the relationship $S_0^2 = S_1^2 + S_2^2 + S_3^2$. Correspondingly, the state of polarization of fully polarized incident laser light used in TS measurements and the evolution of the polarization of monochromatic laser light used for interferometric/polarimetric diagnostics are described by the reduced three-component unit Stokes vector $S_i/S_0\,(i=1,2,3)$. This unit vector is characterized by the azimuth (orientation angle) of the polarization ellipse $0 \le \psi < \pi$ and the ellipticity angle $\chi = \pm\arctan(b_2/b_1)$ determined by the ratio of the minor and the major semi-axis $(-\pi/4 < \chi \le \pi/4)$. We use in this paper the four-component Stokes vector for fully polarized monochromatic incident laser light with arbitrary elliptical polarization given in Equation (5) (see Appendix A 1)

$$\mathbf{S}^{(i)} = E_0^2(1,\ \cos 2\psi\cos 2\chi,\ \sin 2\psi\cos 2\chi,\ \sin 2\chi). \tag{2}$$

A fully unpolarized wave (natural light) is characterized by $S_1 = S_2 = S_3 = 0$. Any partially polarized wave can be decomposed into completely unpolarized and polarized portions. As they are statistically independent, the 4-component Stokes vector of the mixture is a sum of the respective vectors of the separate waves. Defining the unpolarized and polarized parts as $\mathbf{S}^{(unpol)} = (S_0 - \sqrt{S_1^2+S_2^2+S_3^2},\ 0,\ 0,\ 0)$ and $\mathbf{S}^{(pol)} = (\sqrt{S_1^2+S_2^2+S_3^2},\ S_1,\ S_2,\ S_3)$ yields the degrees of polarization/ depolarization of the original wave of intensity S_0[15]

$$P = \frac{I_{pol}}{I_{tot}} = \frac{\sqrt{S_1^2+S_2^2+S_3^2}}{S_0}, \quad D = 1 - P. \tag{3}$$

The analysis of the degree of polarization for Thomson scattering is based on a derivation of the 4×4 Mueller matrix that expresses the Stokes vector of the scattered radiation in terms of the Stokes vector of fully polarized incident laser light. The problem is considered within the scope of classical electrodynamics, where the scattering of the waves is treated as a reemission of electromagnetic radiation by free electrons oscillating in electric and magnetic fields of the incident laser light.

A. Electric field from a single electron

Using the equation for the Lienard-Wiechert potentials, the scattered electric field in the far-zone $\mathbf{E}_s(\mathbf{r},t)$ is expressed by a 2×2 matrix $\hat{\mathbf{\Pi}}$ transforming the incident field to the scattered field in the process of interaction with a single electron moving with velocity $\mathbf{v}$. These calculations are presented in Appendix A 3. The electric fields of the incident and scattered waves are projected, respectively, on the unit vectors $(\mathbf{e}_x,\ \mathbf{t}_i)$ and $(\mathbf{e}_x, \mathbf{t}_s)$ which are orthogonal to the wave propagation directions $\mathbf{i} = \mathbf{k}_i/|\mathbf{k}_i|$ and $\mathbf{s} = \mathbf{k}_s/|\mathbf{k}_s|$

$$\begin{aligned} &\mathbf{E}_i = E_{ix}\mathbf{e}_x + E_{it}\mathbf{t}_i, \quad \mathbf{E}_s = E_{sx}\mathbf{e}_x + E_{st}\mathbf{t}_s, \\ &\mathbf{t}_i = \mathbf{i}\times\mathbf{e}_x = \frac{\mathbf{i}\cos\theta - \mathbf{s}}{\sin\theta}, \quad \mathbf{t}_s = \mathbf{s}\times\mathbf{e}_x = \frac{\mathbf{i} - \mathbf{s}\cos\theta}{\sin\theta}, \end{aligned} \tag{4}$$

where θ is the scattering angle in the scattering plane determined by the vectors $\mathbf{i}$ and $\mathbf{s}$ while the unit vector $\mathbf{e}_x = [\mathbf{i}\times\mathbf{s}]/\sin\theta$ is normal to the scattering plane. The fully polarized incident monochromatic wave is assumed to have an arbitrary elliptical polarization, with semi-major axis b_1 and semi-minor axis b_2, and complex amplitude $\mathbf{E}_i$

$$\begin{aligned} \mathcal{E}_i(\mathbf{r}',t') &= \mathbf{E}_i\exp(i\mathbf{k}_i\cdot\mathbf{r}' - i\omega_i t'), \\ \mathbf{E}_i &= E_0(b_1\mathbf{e}'_x + ib_2\mathbf{e}'_y)/\sqrt{b_1^2+b_2^2}, \end{aligned} \tag{5}$$

where E_0 is the magnitude of the incident wave. The two mutually perpendicular unit vectors $\mathbf{e}'_x$ and $\mathbf{e}'_y$ are orthogonal to the incident wave propagation direction $\mathbf{i}$. Their position with respect to the scattering plane is arbitrary and determined by the azimuth ψ (orientation angle) of the polarization ellipse $(\cos\psi = \mathbf{e}_x\cdot\mathbf{e}'_x)$. The Stokes vector of the incident wave is calculated in the incident wave reference frame $(\mathbf{e}_x, \mathbf{t}_i, \mathbf{i})$ while the Stokes vector of the scattered wave is defined in Equations (A1) in the scattering wave reference frame $(\mathbf{e}_x, \mathbf{t}_s, \mathbf{s})$.

The prime symbol for variables t' and $\mathbf{r}'$ is introduced to indicate the retarded time and electron position inside the scattering volume while the variables t and $\mathbf{r}$ are related to the time at the remote detector (observer) localized at the position $\mathbf{r}$. The radius vector $\mathbf{r}$ connects the origin of the coordinate system chosen somewhere in the center of the scattering volume with the point of observation. At large enough r, one can approximate the distance between an individual electron at the position $\mathbf{r}'(t')$ and the point of observation as $R(t') \simeq r - \mathbf{r}'(t')\cdot\mathbf{s}$. The fields at the point of observation are determined by the position of the electron at the earlier time t' such that $t = t' + R(t')/c$. Differentiating this relation over t and t' yields the relationship for the time interval Δt between arrival at the observer of signals which were emitted by the electron over an interval $\Delta t'$ in the scattering volume

$$\Delta t = (1-\beta_s)\Delta t', \tag{6}$$

where the factor $\beta_s = \mathbf{v}\cdot\mathbf{s}/c$ describes the effect of electron thermal motion. The change of the interval is caused by both the effect of retardation and electron motion toward or away from the observer. This time difference leads to different mean powers emitted by the electron and received by the observer. If the averaged scattered power at the observer is $P^{(observer)}$ then the energy received by the observer during the time interval Δt is $P^{(observer)}\Delta t$. Since the same energy is emitted by the electron during the *time-at-particle* interval $\Delta t'$, the average *time-at-particle* power (see Ref. 2) is different from the *time-at-observer* power

$$P^{(particle)} = P^{(observer)}(1-\beta_s). \tag{7}$$

Instead of using the time-dependent scattered field $\mathbf{E}_s(\mathbf{r},t)$, we follow the standard approach and operate with the Fourier transformed components of the truncated electric field $\mathbf{E}_s^{(T)}(\mathbf{r},\omega)$. The truncated scattered electric field is defined in Appendix A 2. The superscript T indicates parametric dependence on the width T of the truncation interval. Instead of the superscript, the dependence on T will sometimes be shown in the arguments of the function. The use of the truncation method is a substantial element of our approach. Truncation resolves the δ-function singularity in the Fourier transformed Doppler shifted monochromatic electric field of the radiation scattered by a single electron. This allows us to calculate quadratic field combinations without the uncertainty caused by the ambiguous treatment of the square of a δ-function.[13]

The linear relationship between the Fourier image of the truncated scattered field and the amplitude $\mathbf{E}_i$ of the incident field is described by the matrix $\hat{\boldsymbol{\Pi}}$ and amplitude factor $f^{(T)}(\omega, \boldsymbol{\beta})$

$$\mathbf{E}_s^{(T)}(\omega) = f^{(T)}(\omega, \boldsymbol{\beta})\hat{\boldsymbol{\Pi}} \cdot \mathbf{E}_i. \tag{8}$$

The spectral characteristics ω and T and spatial dependence on r are included in $f^{(T)}(\omega, \boldsymbol{\beta})$ defined in (A22) while the matrix $\hat{\boldsymbol{\Pi}}$ is presented in (A18). The explicit form of matrix $\hat{\boldsymbol{\Pi}}$ is obtained by substituting the electric field projections (4) in (A18) and the corresponding transformations (A24) and (A25)

$$\begin{aligned} \begin{pmatrix} E_{sx}^{(T)}(\omega) \\ E_{st}^{(T)}(\omega) \end{pmatrix} &= f^{(T)}(\omega, \boldsymbol{\beta}) \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \begin{pmatrix} E_{ix} \\ E_{it} \end{pmatrix}, \\ c &= \beta_i + \beta_s + \beta_i\beta_s - \cos\theta - \frac{(\beta_i + \beta_s)^2}{1 + \cos\theta}, \\ a &= -(1 - \beta_i)(1 - \beta_s) + \beta_x^2(1 - \cos\theta), \\ b &= \beta_x(1 + \cos\theta - \beta_i - \beta_s)\tan\frac{\theta}{2}. \end{aligned} \tag{9}$$

The matrix form (9) consists of three elements a, b, and c and describes the transformation of the polarization. These three elements are functions of the velocity components $\beta_i = \boldsymbol{\beta} \cdot \mathbf{i}$, $\beta_s = \boldsymbol{\beta} \cdot \mathbf{s}$, and $\beta_x = \boldsymbol{\beta} \cdot \mathbf{e}_x$ and the scattering angle θ where $\boldsymbol{\beta} = \mathbf{v}/c$.

B. Mueller matrix formalism

The matrix representation (9) allows us to construct quadratic combinations of the scattered field components needed for calculation of the Stokes vector of the scattered radiation $\mathbf{S}^{(s)}$. The components of $\mathbf{S}^{(s)}$ are defined by time averaged products (A1) of the electric field components $E_{sx}(t)$ and $E_{st}(t)$. The products of the fields and quadratic combinations of their Fourier images $E_{sx}^{(T)}(\omega)$ and $E_{st}^{(T)}(\omega)$ are related to $\mathbf{S}^{(s)}$ by the set of equations (A10)–(A12). They are presented in Appendix A 2 for the particular case of the zero-component $S_0^{(s)}$. Generalization to all other components is straightforward. The final expression for the Stokes vector $\mathbf{S}^{(s)}$ is obtained by performing two intermediate steps. The first step involves introduction of the auxiliary vector $\mathbf{S}^{(s)}(\omega, T)$ defined by the third equation (A12) generalized from the $S_0^{(s)}(\omega, T)$ case to the three other components. All of them are determined by the relations (A1) where the time dependent electric field components of $\mathbf{E}_s(t)$ are substituted by the corresponding Fourier images of the truncated field $\mathbf{E}_s^{(T)}(\omega)$ while time integration is omitted.

Based on the auxiliary vector $\mathbf{S}^{(s)}(\omega, T)$, the spectral density of the Stokes vector $\mathbf{S}^{(s)}(\omega)$ is defined by the limiting transition $T \to \infty$

$$\mathbf{S}^{(s)}(\omega) = \lim_{T\to\infty} \frac{\mathbf{S}^{(s)}(\omega, T)}{2T}. \tag{10}$$

The zero-component of this vector corresponds to the power spectrum of the scattered radiation. Finally, the full Stokes vector $\mathbf{S}^{(s)}$ is obtained by integrating the spectral density $\mathbf{S}^{(s)}(\omega)$ over the spectrum of the scattered radiation

$$\mathbf{S}^{(s)} = \int_{-\infty}^{+\infty} \mathbf{S}^{(s)}(\omega) d\omega. \tag{11}$$

The auxiliary vector $\mathbf{S}^{(s)}(\omega, T)$ is represented by quadratic combinations of the Fourier images of the scattered electric field. These products are expressed in terms of the quadratic combinations of the incident electric fields by making use of the linear relationships $\mathbf{E}_s^{(T)}(\omega) = f^{(T)}(\omega, \boldsymbol{\beta})\hat{\boldsymbol{\Pi}} \cdot \mathbf{E}_i$ in their explicit form (9). The R.H.S. of the resulting expressions contains the square of the absolute value of the function $f^{(T)}(\omega, \boldsymbol{\beta})$, quadratic combinations of the factors a, b, and c and different quadratic combinations of the $\mathbf{E}_i$ components. Expressing the products of the components of $\mathbf{E}_i$ in terms of the components of $\mathbf{S}^{(i)}$ from Equation (1) allows us to obtain the 4×4 auxiliary Mueller matrix $\hat{\mathbf{M}}^{(single)}(\omega, T)$ caused by scattering on a single electron moving with velocity $\boldsymbol{\beta}$. This matrix connects the auxiliary Stokes vector of the scattered radiation with the Stokes vector of the incident wave

$$\mathbf{S}^{(s)}(\omega, T) = \hat{\mathbf{M}}^{(single)}(\omega, T) \cdot \mathbf{S}^{(i)}. \tag{12}$$

It is useful to present the auxiliary Mueller matrix as a product $\hat{\mathbf{M}}^{(single)}(\omega, T) = C^{(T)}(\omega)\hat{\mathbf{W}}(\boldsymbol{\beta})$, where the scalar function $C^{(T)}(\omega)$ is proportional to the square of the absolute value of the function $f^{(T)}(\omega, \boldsymbol{\beta})$ (A22)

$$C^{(T)}(\omega) = \frac{r_0^2(1 - \beta^2)E_0^2}{2r^2(1 - \beta_s)^6}\left[\frac{2}{\pi}\frac{\sin^2(\omega - \omega_d)T}{(\omega - \omega_d)^2}\right], \tag{13}$$

while Doppler shifted frequency of the scattered radiation $\omega_d = \omega_i(1 - \beta_i)/(1 - \beta_s))$. The 4×4 matrix $\hat{\mathbf{W}}(\boldsymbol{\beta})$ is expressed by quadratic combinations of the coefficients a, b, and c

$$\hat{\mathbf{W}}(\boldsymbol{\beta}) = \begin{pmatrix} a^2 + 2b^2 + c^2 & a^2 - c^2 & 2b(a - c) & 0 \\ a^2 - c^2 & a^2 - 2b^2 + c^2 & 2b(a + c) & 0 \\ 2b(c - a) & -2b(a + c) & 2(ac - b^2) & 0 \\ 0 & 0 & 0 & 2(b^2 + ac) \end{pmatrix}. \tag{14}$$

The factor E_0^2 is included in (13) from expression (2) for the Stokes vector $\mathbf{S}^{(i)}$ of the incident wave. Correspondingly, in all further equations $\mathbf{S}^{(i)}$ is treated as a dimensionless normalized vector (2) without the E_0^2 factor, $\mathbf{S}^{(i)} \to \mathbf{S}^{(i)}/E_0^2$. The renormalized vector $\mathbf{S}^{(i)}$ describes the dependence of the scattered radiation on the polarization characteristics of the incident light determined by the orientation ψ and ellipticity χ angles. The factor 1/2 in (13) originates from the similar factor in the R.H.S. of (1).

Substituting Equation (12) into (10) and performing limiting transition yields the spectral density of the Stokes vector $\mathbf{S}^{(s)}(\omega)$ as a product of the spectral Mueller matrix $\hat{\mathbf{M}}^{(single)}(\omega)$ and $\mathbf{S}^{(i)}$

$$\mathbf{S}^{(s)}(\omega) = \hat{\mathbf{M}}^{(single)}(\omega) \cdot \mathbf{S}^{(i)}. \quad (15)$$

The limiting transition modifies the scalar function (13) but does not affect $\hat{\mathbf{W}}$ so that the spectral Mueller matrix takes a form

$$\begin{aligned} \hat{\mathbf{M}}^{(single)}(\omega) &= C(\omega)\hat{\mathbf{W}}(\boldsymbol{\beta}), \\ C(\omega) &= \frac{r_0^2\left(1-\beta^2\right)E_0^2}{2r^2\left(1-\beta_s\right)^6}\delta(\omega-\omega_d). \end{aligned} \quad (16)$$

The specific form of the δ-function (16) is rigorously determined by the limiting transition $T \to \infty$ without the uncertainty associated with the phenomenological "recipe" for treatment of the square of a δ-function (see (A13)–(A15)).

According to (11), the full Stokes vector of the scattered radiation $\mathbf{S}^{(s)}$ is determined by integrating $\mathbf{S}^{(s)}(\omega)$ over the entire frequency spectrum. This yields $\mathbf{S}^{(s)}$ as a product of the frequency integrated Mueller matrix $\hat{\mathbf{M}}^{(single)}$ and $\mathbf{S}^{(i)}$

$$\mathbf{S}^{(s)} = \hat{\mathbf{M}}^{(single)} \cdot \mathbf{S}^{(i)}. \quad (17)$$

Explicit integration over ω in (16) removes the δ-function dependence in $C(\omega)$ and yields the frequency integrated Mueller matrix

$$\hat{\mathbf{M}}^{(single)} = C\hat{\mathbf{W}}(\boldsymbol{\beta}), \quad C = \frac{r_0^2\left(1-\beta^2\right)E_0^2}{2r^2\left(1-\beta_s\right)^6} \quad (18)$$

that describes the transformation of the Stokes vector caused by scattering on a single electron moving with velocity $\boldsymbol{\beta}$. All three Mueller matrices describe linear connections of the corresponding Stokes vectors of the scattered radiation with the incident Stokes vector $\mathbf{S}^{(i)}$. They are almost identical in structure with different amplitude factors $C^{(T)}(\omega)$, $C(\omega)$, or C while the fundamental matrix $\hat{\mathbf{W}}(\boldsymbol{\beta})$ is the same in all cases.

The procedure of integration over the spectrum corresponds to a transition from the spectrum-based characteristics to the polarization analysis based on the total frequency integrated Stokes vector spectral intensities. Integrating over all frequencies results in an increased number of detected photons with better statistics and accuracy of measurements. This is a key element of the polarization-based TS diagnostic compared to the traditional spectrum-based TS method. Since the purpose of our work is to investigate the optimal capabilities of depolarization diagnostics, we will focus on the characteristics of the total scattered radiation.

The velocity $\boldsymbol{\beta}$ as well as the polarization parameters ψ, χ of the incident wave are arbitrary in Equations (17) and (18). This allows us to test the single electron Mueller matrix $\hat{\mathbf{M}}^{(single)}$ by comparing with known solutions. One such example is the solution to problem 6 in Sec. 78 in Ref. 16. It represents the angular distribution of the scattering power for a linearly polarized incident wave scattered by a charge moving with velocity $\boldsymbol{\beta}$ in the direction of propagation of the incident wave. In this particular case, the fully relativistic acceleration is perpendicular to the velocity yielding a relatively simple expression for the scattering cross-section

$$\begin{aligned} d\sigma = \left(\frac{e^2}{m_e c^2}\right)^2 \frac{\left(1-\beta^2\right)(1-\beta)^2}{(1-\beta\sin\Theta\cos\Phi)^6} \\ \times\left[(1-\beta\sin\Theta\cos\Phi)^2-\left(1-\beta^2\right)\cos^2\Theta\right]d\Omega, \end{aligned} \quad (19)$$

where $d\sigma$ is the ratio of the energy scattered into the solid angle $d\Omega$ per unit time to the energy flux density of the incident radiation. The propagation direction $\mathbf{s}$ is characterized in Ref. 16 by the polar and azimuthal angles Θ, Φ relative to a spherical coordinate system with z-axis along $\mathbf{E}_i$ and x-axis along $\boldsymbol{\beta}$. Putting $\chi = 0$ for a linear polarized incident wave allows us to express the variables θ and ψ in terms of Θ, Φ (see (A2) and (A4))

$$\cos\theta = \sin\Theta\cos\Phi, \quad \cos^2\psi = \frac{\sin^2\Theta\sin^2\Phi}{1-\sin^2\Theta\cos^2\Phi} \quad (20)$$

and to obtain the total scattering power $S_0^{(s)}$ given in (17) and (18) in terms of the variables Θ and Φ. Calculating a, b, and c factors in $\hat{\mathbf{W}}$ by putting $\beta_i = \beta$, $\beta_s = \beta\cos\theta$, $\beta_x = 0$ yields $S_0^{(s)}$ and the corresponding cross-section which is identical to the solution (19) increasing confidence in correctness of Equations (16) and (18).

All three variants of the Mueller matrix correspond to scattering on a single electron moving with velocity $\mathbf{v}$. Scattering from a single electron changes the frequency and polarization, but the scattered wave continues to be monochromatic and, therefore, fully polarized. Correspondingly, all Mueller matrices conserve polarization and transfer fully polarized incident light to fully polarized scattered radiation for an arbitrary electron velocity

$$\begin{aligned} &S_0^{(s)2} - S_1^{(s)2} - S_2^{(s)2} - S_3^{(s)2} \\ &\quad = \left(b^2+ac\right)^2 C^2\left(S_0^{(i)2} - S_1^{(i)2} - S_2^{(i)2} - S_3^{(i)2}\right) = 0. \end{aligned}$$

III. COMBINED EFFECT OF MANY ELECTRONS

Equations (16) and (18) describe the elementary process of scattering on an individual electron moving in unbounded space filled with an incident homogeneous plane electromagnetic wave of infinite extent. They are used now to account for scattering from many electrons. We illustrate the calculations for the zero-component of the Stokes vector and justify the approach for the three other components.

The zero-component represents the power spectrum of the scattered radiation at the remote detector

$$P^{(single)}(\omega) = \frac{r_0^2\left(1-\beta^2\right)E_0^2}{2r^2\left(1-\beta_s\right)^6}\left(\hat{\mathbf{W}}(\boldsymbol{\beta}) \cdot \mathbf{S}^{(i)}\right)_0 \delta(\omega-\omega_d), \quad (21)$$

where the non-zero tensor components W_{00}, W_{01}, and W_{02} are expressed in terms of a, b, and c in (14) while explicit dependences of these factors on $\boldsymbol{\beta}$ are given in (9) or, equivalently, in (27) and (28). Although the cross section for Thomson scattering is small (proportional to r_0^2), the intensity of the scattered radiation is measurable due to the large number of electrons $N \gg 1$ participating in scattering. The total electric field of the scattered radiation is a sum of the electric fields emitted by the individual electrons. The coherency matrix is constructed from time-averaged quadratic combinations of the electric field components. The products of the field components are subdivided into two groups. There is a large number $\propto N^2$ of cross-terms originating from the electrons characterized by different positions $\mathbf{R}_0^{(i)}$ and $\mathbf{R}_0^{(j)}$ with $i \neq j$ where the vectors $\mathbf{R}_0^{(i)}$ $(i = 1, 2, ...N)$ are introduced in (A16) and serve as labels of the unperturbed electron trajectories. Summing over many electrons, we assume the condition of incoherent Thomson scattering $\lambda_D|\boldsymbol{q}|\sin\theta/2 \gg 1$, where the Debye length λ_D represents the mean spatial electron correlation length. The regime of incoherent scattering will be realized in ITER for a conventional TS diagnostic with laser wavelength $\lambda = 1\mu$m and $\theta \simeq 130°$. Collective TS regimes with large wavelengths or small scattering angles are used for measurements of the bulk and fast ion characteristics (see a detailed review in Ref. 17). In incoherent regime, the cross-terms are proportional to rapidly oscillating factors $\exp[-i\mathbf{q}\cdot(\mathbf{R}_0^{(i)} - \mathbf{R}_0^{(j)})]$ with $\mathbf{q} = \mathbf{k}_s - \mathbf{k}_i$ and, therefore, vanish after summation. Then, the products of the sum of the electric fields are reduced to the sum of the products characterized by equal indices $i = j$. Thus, for incoherent TS, the Stokes vector of the scattered radiation is the sum of the Stokes vectors of the radiation scattered by the individual electrons. The summation of these $\boldsymbol{\beta}$-dependent quantities is equivalent to integration over $d\mathbf{r}$ and $d\boldsymbol{\beta}$ in coordinate and velocity space. According to the time averaging variable t in (A1), the area of the $d\mathbf{r}$-integration should correspond to the summation over those electrons whose pulses of the scattered radiation are passing through the detector at a given time t on the detector. This determines the area of the $d\mathbf{r}$-integration over the modified scattering volume (see Appendix B) that results in the FTT expression (24) used below.

The equilibrium electron distribution function is defined as the number of electrons $dN = n_e f_M(\boldsymbol{\beta})\, d\boldsymbol{\beta}\, d\mathbf{r}$ with velocities in the interval $\boldsymbol{\beta}$, $\boldsymbol{\beta} + d\boldsymbol{\beta}$ contained in a volume element $d\mathbf{r}$, where $f_M(\boldsymbol{\beta})$ is the relativistic Maxwellian distribution function normalized to unity

$$f_M(\boldsymbol{\beta}) = \frac{\mu \exp\left(-\mu/\sqrt{1-\beta^2}\right)}{4\pi K_2(\mu)\left(1-\beta^2\right)^{5/2}}, \quad \mu = m_e c^2/T_e, \tag{22}$$

and $K_2(\mu)$ is the modified Bessel function of the second kind.[18] We first select a group of electrons having equal velocities $\boldsymbol{\beta}$ in the velocity element $d\boldsymbol{\beta}$ but different initial positions $\mathbf{R}_0$ in coordinate space. The spectral powers (21) registered by the detector from each of these electrons are the same (do not depend on $\mathbf{R}_0$). The intuitive way of accounting for the effect of many electrons is to multiply the power from a single electron $P^{(single)}(\omega)$ by the total number of electrons $n_e V f_M d\boldsymbol{\beta}$ in the scattering volume V. The resulting total spectral power

$$P_{ITT}(\omega) = \frac{r_0^2 n_e V\left(1-\beta^2\right)E_0^2 f_M d\boldsymbol{\beta}}{2r^2(1-\beta_s)^6}\left(\hat{\mathbf{W}}(\boldsymbol{\beta})\cdot\mathbf{S}^{(i)}\right)_0 \delta(\omega-\omega_d) \tag{23}$$

is equivalent to Equation (10) derived in the first part of Ref. 12 (Sec. II C) devoted to the infinite scattering volume or infinite transit time (ITT) case.

This intuitive approach can fail to accurately characterize the scattered power. As was pointed out by Stupakov,[19] a more consistent approach is not a summation of the instantaneous powers but a summation of energies emitted by the electrons and accumulated by the detector during some time interval long compared to the particle transit time through the scattering volume. The problem was formally treated in the second part of Ref. 12 (Sec. II D) devoted to the finite scattering volume or, equivalently, to the finite transit time (FTT) case by applying the Fourier transform in coordinate space leading to the result

$$P_{FTT}(\omega) = \frac{r_0^2 n_e V\left(1-\beta^2\right)E_0^2 f_M d\boldsymbol{\beta}}{2r^2(1-\beta_s)^5}\left(\hat{\mathbf{W}}(\boldsymbol{\beta})\cdot\mathbf{S}^{(i)}\right)_0 \delta(\omega-\omega_d). \tag{24}$$

The only difference between the ITT power spectrum (23) and the FTT case (24) is an additional factor $(1-\beta_s)$ in the numerator of the FTT intensity spectrum. The FTT weighting factor is generally accepted in all present-day relativistic treatments of Thomson scattered radiation. Some details related to the FTT effect are briefly discussed in Appendix B. Based on the FTT form (24), we now perform the final stage of the summation by averaging the scattered Stokes vector over velocities $\boldsymbol{\beta}$.

IV. MUELLER MATRIX FOR INCOHERENT THOMSON SCATTERING

A. Averaging over β with the FTT weighting factor

Averaging over velocity space is performed by integrating the Mueller matrix (18) over the relativistic Maxwellian distribution function (22). The combined effect of many electrons and finite size of the scattering volume are taken into account by adding the total number of electrons in the scattering volume $N = n_e V$ and the factor $(1-\beta_s)$ to the scattering operator. Both these factors were missed in Ref. 8. The resulting expression has the form

$$\hat{\mathbf{M}}(\mu,\theta) = \frac{r_0^2 N E_0^2}{2r^2}\int\frac{(1-\beta^2)f_M(\boldsymbol{\beta})d\boldsymbol{\beta}}{(1-\beta_s)^5}\hat{\mathbf{W}}(\boldsymbol{\beta}). \tag{25}$$

For compact notation, it is suitable to operate with the normalized dimensionless matrix $\hat{\mathbf{m}}(\mu,\theta)$

$$\hat{\mathbf{m}}(\mu,\theta) = \hat{\mathbf{M}}(\mu,\theta)/C_0, \tag{26}$$

where the factor $C_0 = r_0^2 N E_0^2/2r^2$.

The integration (25) is performed in spherical coordinates with the v_z and v_x axis directed along $\mathbf{s}$ and $\mathbf{e}_x$, respectively, such that $\boldsymbol{\beta} = (\beta \sin\alpha \cos\phi, \beta \sin\alpha \sin\phi, \beta\cos\alpha)$, where α is the polar angle and ϕ is the azimuthal angle measured from $\mathbf{e}_x$ in the orthogonal plane. The factors a, b, and c in expression (14) for $\hat{\mathbf{W}}(\boldsymbol{\beta})$ are defined in Equation (9). They depend on three non-orthogonal components of the electron velocity: $\beta_i = \boldsymbol{\beta}\cdot\mathbf{i}$, $\beta_s = \boldsymbol{\beta}\cdot\mathbf{s}$, and $\beta_x = \boldsymbol{\beta}\cdot\mathbf{e}_x$, where $\boldsymbol{\beta} = \mathbf{v}/c$. The vector $\boldsymbol{\beta}$ is characterized by its spherical coordinates, or by its Cartesian coordinates $\beta_x, \beta_y, \beta_z$. The non-orthogonal projections can be expressed in terms of Cartesian coordinates or, equivalently, as functions of the polar angle α and azimuthal angle ϕ

$$\begin{aligned}&\beta_x = \beta\sin\alpha\cos\phi, \quad \beta_s = \beta_z = \beta\cos\alpha\\ &\beta_i = \beta_y\sin\theta + \beta_z\cos\theta = \beta(\sin\alpha\sin\phi\sin\theta + \cos\alpha\cos\theta).\end{aligned} \tag{27}$$

Substituting (27) in (9) and taking into account partial cancelations in the terms proportional to $(1+\cos\theta)^{-1}$ and $\tan(\theta/2)$ yields a, b, and c as functions of $u = \cos\theta$, $x = \cos\alpha$, and ϕ

$$\begin{aligned}a &= Q\sin\phi + \beta^2(1-u)(1-x^2)\cos^2\phi - ux^2\beta^2 + ux\beta + x\beta - 1,\\ b &= Q\cos\phi - \beta^2(1-u)(1-x^2)\sin\phi\cos\phi,\\ c &= Q\sin\phi - \beta^2(1-u)(1-x^2)\sin^2\phi + (x\beta-1)(u-x\beta),\end{aligned} \tag{28}$$

where the factor Q is introduced to combine the dependences of a, b and c on $\sin\alpha$ and $\sin\theta$

$$Q = \beta(1-x\beta)\sqrt{1-u^2}\sqrt{1-x^2}. \tag{29}$$

Averaging over the relativistic Maxwellian distribution function (25) consists of three successive integrations

$$\begin{aligned}\hat{\mathbf{m}}(\mu,\theta) = &\frac{\mu}{4\pi K_2(\mu)}\int_0^1 \frac{\beta^2\exp\left(-\mu/\sqrt{1-\beta^2}\right)d\beta}{\left(1-\beta^2\right)^{3/2}}\\ &\times\int_0^\pi \frac{\sin\alpha d\alpha}{(1-\beta\cos\alpha)^5}\int_0^{2\pi} d\phi\hat{\mathbf{W}}(\beta,\alpha,\phi).\end{aligned} \tag{30}$$

Four elements of the matrix $\hat{\mathbf{W}}$ are proportional to $b \propto \beta_x$. They average to zero after integration over the velocity space. In order to prove this property, let us consider integration in Cartesian coordinates with $d\boldsymbol{\beta} = d\beta_x d\beta_y d\beta_z$. From (9) and (27), it follows that factor c does not depend on β_x while factor a is even and factor b is odd in β_x. From the Mueller matrix, it follows that the four elements W_{02}, W_{12}, W_{20}, W_{21} are odd functions of β_x. Since the Maxwellian distribution function as well as all other weighting factors are even functions of β_x, these four elements cancel after averaging. The remaining five elements are integrated according to Equation (30). As a first step, we integrate over the azimuthal angle ϕ

$$\hat{\mathbf{m}}^{(\phi)} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\hat{\mathbf{W}}(\theta,\beta,\alpha,\phi). \tag{31}$$

The ϕ-dependences of the matrix elements are determined by quadratic combinations of the factors (28) and are represented by the products of different powers of the trigonometric functions $\sin\phi$ and $\cos\phi$. Analytical integration of these combinations over ϕ is straightforward and leads to the following results:

$$\begin{aligned}m_{00}^{(\phi)} &= \beta^4(3u^2-1)x^4 - 2\beta^3(u+1)(3u-1)x^3 + \beta^2\left((1+2u-3u^2)\beta^2 + 1 + 6u + 5u^2\right)x^2\\ &\quad -2\beta\left(2\beta^2(1-u^2) + (u+1)^2\right)x + 1 + u^2 + \beta^4(u-1)^2 + \beta^2(1+2u-3u^2)\\ m_{01}^{(\phi)} &= m_{10}^{(\phi)} = \left(1-\beta^2\right)(1-u^2)(1-x\beta)^2\\ m_{02}^{(\phi)} &= m_{12}^{(\phi)} = m_{20}^{(\phi)} = m_{21}^{(\phi)} = 0\\ m_{11}^{(\phi)} &= \frac{1}{2}\beta^4(u+1)^2x^4 - 2\beta^3(u+1)^2x^3 + 3\beta^2(u+1)^2x^2 - 2\beta(u+1)^2x + 1 + u^2 + \frac{1}{2}\beta^4(u-1)^2 - \beta^2(u-1)^2\\ m_{22}^{(\phi)} &= \frac{1}{2}\beta^4(u+1)^2x^4 - 2\beta^3(u+1)^2x^3 + 3\beta^2(u+1)^2x^2 - 2\beta(u+1)^2x + \frac{1}{2}\left(\beta^2(u-1)+2\right)\left(\beta^2(1-u)+2u\right)\\ m_{33}^{(\phi)} &= \beta^4(3u^2-1)x^4 - 2\beta^3(u+1)(3u-1)x^3 + \beta^2\left((1+2u-3u^2)\beta^2 + 1 + 6u + 5u^2\right)x^2\\ &\quad -2\beta\left(2\beta^2(1-u^2) + (u+1)^2\right)x + 2u + \beta^2(1-u)(u+3).\end{aligned} \tag{32}$$

These expressions are quadratic polynomial functions of $u = \cos\theta$, fourth degree polynomials of $x = \cos\alpha$ and do not contain terms proportional to $\sin\alpha = \sqrt{1-x^2}$ and $\sin\theta$. Cancellation of the $\sin\alpha$ and $\sin\theta$ terms is critical for exact analytical calculations. Indeed, it is seen from (28) that these two factors are combined in one variable Q which is multiplied by either $\sin\phi$ or $\cos\phi$. Forming quadratic combinations of a, b, and c according to Equation (14) for $\hat{\mathbf{W}}$ shows that the terms linear in Q are proportional to the first powers of $\sin\phi$ or $\cos\phi$ which average to zero after integration over ϕ. This results in polynomial functions (32) of x and u without contributions that are linear in $\sin\alpha$ and $\sin\theta$.

Averaging the matrix elements (32) over α is equivalent to integration over x

$$\hat{\mathbf{m}}^{(\alpha)} = \frac{1}{2}\int_{-1}^{1} \frac{dx\hat{\mathbf{m}}^{(\phi)}(x,\beta,u)}{(1-\beta x)^5}. \tag{33}$$

Because of polynomial dependence on x, the corresponding integrals are of the form

$$I_n = \frac{1}{2}\int_{-1}^{1} \frac{x^n dx}{(1-\beta x)^5}, \quad n = 0, 1, ..., 4. \tag{34}$$

They are evaluated analytically by successive integrations by parts. The results are represented by rational functions of β for $0 \le n \le 3$ and by a hyperbolic arc-tangent function for $n = 4$

$$I_0 = \frac{1+\beta^2}{(1-\beta^2)^4},\ I_1 = \frac{\beta(\beta^2+5)}{3(1-\beta^2)^4},\ I_2 = \frac{5\beta^2+1}{3(1-\beta^2)^4},$$
$$I_3 = \frac{\beta(1+\beta^2)}{(1-\beta^2)^4},\ I_4 = \frac{(12\beta^6-14\beta^4+11\beta^2-3)}{3\beta^4(\beta^2-1)^4} + \frac{\tanh^{-1}(\beta)}{\beta^5}. \tag{35}$$

Performing integration (33) over x yields

$$m_{00}^{(\alpha)} = \frac{(3u^2-1)\tanh^{-1}(\beta)}{\beta} + \frac{2(u-1)(\beta^2+(5\beta^2-3)u-3)}{3(\beta^2-1)^2},$$
$$m_{01}^{(\alpha)} = m_{01}^{(\alpha)} = \frac{u^2-1}{\beta^2-1},$$
$$m_{11}^{(\alpha)} = \frac{1}{2}\left(\frac{(\beta^2+1)(u-1)^2}{(\beta^2-1)^2} + \frac{(u+1)^2\tanh^{-1}(\beta)}{\beta}\right),$$
$$m_{22}^{(\alpha)} = \frac{1}{2}\left(\frac{(u+1)^2\tanh^{-1}(\beta)}{\beta} - \frac{(\beta^2+1)(u-1)^2}{(\beta^2-1)^2}\right),$$
$$m_{33}^{(\alpha)} = \frac{(3u^2-1)\tanh^{-1}(\beta)}{\beta} + \frac{(u-1)(5\beta^2+(7\beta^2-9)u-3)}{3(\beta^2-1)^2}. \tag{36}$$

Integration of $\hat{\mathbf{m}}^{(\alpha)}$ over the absolute value of the electron velocity β is performed by substituting the variable of integration in (30), $\beta = 1 - 1/\gamma^2$

$$\mathbf{m}(\mu,\theta) = \frac{\mu}{K_2(\mu)}\int_1^\infty \frac{d\gamma}{\gamma}\exp(-\mu\gamma)\sqrt{\gamma^2-1}\,\mathbf{m}^{(\alpha)}(\gamma,u). \tag{37}$$

Expressions (36) contain three different rational functions of β proportional to $(1-\beta^2)^{-1} = \gamma^2$, $(1-\beta^2)^{-2} = \gamma^4$, and $\beta^2/(1-\beta^2)^2 = \gamma^2(\gamma^2-1)$ and the function $\tanh^{-1}(\beta)/\beta$. Integration of the first three combinations yields

$$\int_1^\infty d\gamma\gamma(\gamma^2-1)^{1/2}\exp(-\mu\gamma) = \frac{K_2(\mu)}{\mu},$$
$$\int_1^\infty d\gamma\gamma^3(\gamma^2-1)^{1/2}\exp(-\mu\gamma) = \frac{K_2(\mu)}{\mu} + \frac{3K_3(\mu)}{\mu^2}, \tag{38}$$
$$\int_1^\infty d\gamma\gamma(\gamma^2-1)^{3/2}\exp(-\mu\gamma) = \frac{3K_3(\mu)}{\mu^2}.$$

They follow from the integral representation for modified Bessel functions of the second kind $K_n(\mu)$[18]

$$K_n(\mu) = \frac{\sqrt{\pi}\mu^n}{2^n\Gamma\left(n+\frac{1}{2}\right)}\int_1^\infty d\gamma\exp(-\mu\gamma)(\gamma^2-1)^{n-\frac{1}{2}}. \tag{39}$$

In order to transform the integrals (38) to the canonical form (39), we regroup the terms and perform integration by parts to get rid of powers of γ in the integrands. The fourth combination with the hyperbolic arc-tangent function leads to the following integral, which is converted to the canonical form (39) by integration by parts:

$$\int_1^\infty d\gamma\exp(-\mu\gamma)\tanh^{-1}(\beta) = \frac{1}{\mu}\int_1^\infty \frac{d\gamma\exp(-\mu\gamma)}{\sqrt{\gamma^2-1}} = \frac{K_0(\mu)}{\mu}, \tag{40}$$

where the derivatives resulting from integration by parts are as follows: $d\tanh^{-1}(\beta)/d\beta = \gamma^2$ and $d\beta/d\gamma = \gamma^{-2}(\gamma^2-1)^{-1/2}$. Expressions (38) and (40) contain four different modified Bessel functions. They are reduced to two functions $K_{1,2}(\mu)$ by making use of the recurrence relations,[18] $K_0(\mu) = K_2(\mu) - 2K_1(\mu)/\mu,\ K_3(\mu) = K_1(\mu) + 4K_2(\mu)/\mu$.

Evaluating integral (37) with the use of these relationships gives the final presentation for the elements of the Mueller $\hat{\mathbf{m}}(\mu,\theta)$. All integrations are performed in analytical form yielding functions of the scattering angle, $u = \cos\theta$, and electron temperature via the factor μ^2 and function $G(\mu) = K_1(\mu)/(\mu K_2(\mu))$, where K_1 and K_2 are modified Bessel functions of the second kind

$$m_{00} = 1 + u^2 - 2G(\mu)(u^2+4u-3) + (16/\mu^2)(1-u)^2,$$
$$m_{01} = m_{10} = 1 - u^2,$$
$$m_{11} = 1 + u^2 + 2G(\mu)(u^2-4u+1) + (12/\mu^2)(1-u)^2,$$
$$m_{22} = 2u - 4G(\mu)(u^2-u+1) - (12/\mu^2)(1-u)^2,$$
$$m_{33} = 2u - 4G(\mu)u(2u-1) - (8/\mu^2)(1-u)^2. \tag{41}$$

The matrix elements (41) present an exact analytical solution for the state of polarization of incoherent Thomson scattering radiation. In contrast to Ref. 8 where only the lowest order linear in T_e analytical results were obtained on the basis of the incorrect ITT weighting factor, expressions (41) are valid for the full range of scattering angles and electron thermal motion from non-relativistic to ultra-relativistic. The first terms in (41) describe the change of polarization in cold plasma ($\mu \to \infty$), the second terms yield first order corrections in the weakly relativistic limit at $\mu \gg 1$, and the third terms dominate at ultra-relativistic temperatures $\mu \ll 1$.

The Mueller matrix $\hat{\mathbf{m}}(\mu,\theta)$ does not conserve polarization and transfers fully polarized incident light to partially polarized scattered radiation. This property is intrinsically connected with the broadening of the scattered spectrum, which by definition is no longer fully polarized. The degree of polarization P is defined in Equation (3). It is a ratio of power flux in the polarized component to the total power flux. Since P is a ratio of two fluxes, the normalization factor C_0 cancels in the final expression for P. Thus, the degree of polarization/depolarization is completely determined by the elements of the matrix $\hat{\mathbf{m}}(\mu,\theta)$ and the Stokes vector of the incident radiation $\mathbf{S}^{(i)}(\psi, \chi)$. It depends on the characteristics of the incident light ψ and χ, electron temperature $T_e = m_e c^2/\mu$ and the scattering angle θ. Detailed information about the properties of this function of four variables is presented in Ref. 10.

B. Mueller matrix averaged over β with the ITT weighting factor

The above results are obtained by averaging with the weighting factor $\propto (1-\beta^2)/(1-\beta_s)^5$. The technique of analytical integration over $\boldsymbol{\beta}$ can also be applied to the ITT weighting factor treated in Ref. 8. This also yields Mueller matrix elements valid at all temperatures. Although the sixth power scaling seems to be irrelevant for TS applications, it is useful for the purpose of comparison to illustrate the importance of the specific form of the weighting factor. The sixth power weighting factor is expressed by the Mueller matrix

$$\hat{\mathbf{H}}(\mu, \theta) = C_0 \hat{\mathbf{h}}(\mu, \theta) \tag{42}$$

defined by the integral (25) with the factor $(1-\beta_s)^6$ in the denominator. The results of exact calculation of the matrix $\hat{\mathbf{h}}$ are given in Equation (B1) in Appendix B. Comparison of the two Mueller matrices $\hat{\mathbf{m}}$ and $\hat{\mathbf{h}}$ reveals essential differences. For example, the off-diagonal elements $m_{01} = m_{10}$ do not depend on electron temperature while the same elements of the matrix $\hat{\mathbf{h}}$ are substantial functions of T_e. The temperature independence of the off-diagonal elements $m_{01} = m_{10}$ is a unique consequence of the fifth power weighting factor. The same integration performed for any other weighting factor results in temperature dependent off-diagonal elements.

A good test of the exact analytical calculations is comparison with the first-order expansions in T_e presented in Equation (44) in Ref. 8. Similar approximation can be obtained from the exact matrix $\hat{\mathbf{H}}(\mu, \theta)$ by Taylor expansion of the matrix element (B1) in powers of small electron temperature $T_e/m_e c^2 = 1/\mu \ll 1$ $(\mu \to \infty)$. Ignoring small terms proportional to μ^{-2} and μ^{-4} and taking into account that $G(\mu) \to 1/\mu = T_e/m_e c^2$ yields the first order correction in T_e to the cold plasma Mueller matrix. Comparing this correction with expressions (44) in Ref. 8 at $\overline{\beta^2} = 3T_e/m_e c^2$ shows that our results are identical. This verifies the first-order expansions in T_e obtained in Ref. 8 for the ITT model and increases confidence in the correctness of our exact analytical scheme of integration.

V. SUMMARY AND FUTURE WORK

The classical problem of depolarization of incoherent Thomson scattered radiation is analytically solved, for the first time, without any approximations. Special attention is paid to justification of the fifth power weighting factor for averaging over the relativistic Maxwellian distribution function. The exponent of the function $(1-\beta_s)$ is important for exact relativistic calculations. This is illustrated by comparison with the case of sixth power weighting factor considered in Ref. 8.

The Mueller matrix averaged over electron thermal motion does not conserve polarization and transfers fully polarized incident light to partially polarized scattered radiation. This property is intrinsically connected with the broadening of the scattered spectrum, which, by definition, is no longer fully polarized. If the degree of polarization dependence on electron temperature is accurately known from theory, the accuracy of such a diagnostic could potentially exceed that of the conventional spectrum-based TS method. Since the scattered spectra are broad for fusion-grade plasmas, all results for the degree of polarization are obtained for the frequency integrated components of the Stokes vector $\mathbf{S}^{(s)}$. Integrating over the spectrum results in an increased number of detected photons with better statistics and measurements accuracy. This is a key element of the polarization-based TS diagnostic compared to the traditional spectrum-based TS method.

The degree of polarization P depends on T_e, the scattering angle θ, and the polarization characteristics of the incident light ψ and χ. In spite of the large number of variables and complexity of the dependencies, the exact analytical results allow us to describe rigorously in a compact form the general properties of the degree of polarization (see Ref. 10). At given θ and T_e, extrema of P as a function of ψ and χ are reached at the boundaries of the region $0 \le \psi \le \pi/2$, $0 \le \chi \le \pi/4$. This yields the absolute maximum $P_{max}(T_e, \theta)$, and minimum $P_{min}(T_e, \theta)$, with respect to all possible polarization states of the incident radiation, and allows us to set upper and lower limits on P at a given θ and T_e. A good test of the correctness of the matrix elements $\hat{\mathbf{m}}$ is that $P < 1$ for all values of the variables. These results are directly used for optimization of the polarization-based TS method of electron temperature measurement in Ref. 11.

For LIDAR TS systems with backscattered detectors at $\theta \sim 180°$, the theoretical model predicts a maximum of depolarization, $D = 1 - P$, of order $3\% - 8\%$ for a circularly polarized laser beam. Because of the quadratic dependence on $T_e/m_e c^2$, the effect is about five times smaller than for perpendicular scattering and is, therefore, difficult to exploit for T_e determination in LIDAR. For a conventional TS system with $\theta \simeq 90°$, the situation is much more favorable with average $D \sim 20\% - 25\%$. The absolute maximum $D_{max} \sim 95\%$ is reached at $\psi = 90°$ for elliptically polarized incident light. This extreme regime corresponds to very small scattered power and results in large error bars for polarization-based T_e measurements. More practical cases with circular and linear incident polarizations are analyzed in Refs. 10 and 11 for conventional TS diagnostics. Although circular incident polarization yields stronger depolarization of scattered radiation, rigorous minimization of the error bars shows that linear incident polarization is preferential for polarization-based diagnostics.

Realistic experimental constraints require detecting scattered photons within a limited wavelength range. This necessitates understanding the frequency resolved degree of polarization.[8] Publications devoted to the rigorous analysis of the spectral polarization concept are not available in the literature. Some empirical steps in this direction can be made on the basis of Equation (11). Let us consider a partial contribution $\Delta\mathbf{S}^{(s)}$ to the spectrum integrated Stokes vector from a narrow frequency interval $\Delta\omega$, such that $\Delta\mathbf{S}^{(s)} = \mathbf{S}^{(s)}(\omega)\Delta\omega$. Formal substitution of $\Delta\mathbf{S}^{(s)}$ to the equation for the degree of polarization (3) yields an expression for $P(\omega)$ which is determined by the spectral density $\mathbf{S}^{(s)}(\omega)$ and does not depended on $\Delta\omega$

$$P(\omega) = \frac{\sqrt{S_1^{(s)2}(\omega) + S_2^{(s)2}(\omega) + S_3^{(s)2}(\omega)}}{S_0^{(s)}(\omega)}. \tag{43}$$

The Stokes vector components $\mathbf{S}^{(s)}(\omega)$ are determined by the single electron spectral Mueller matrix (16) averaged over the relativistic Maxwellian distribution function with the weighting factor (25) and δ-function spectral dependence

$$\hat{\mathbf{M}}(\omega,\mu,\theta) = \frac{r_0^2 n_e V E_0^2}{2r^2}\int \frac{(1-\beta^2) f_M(\boldsymbol{\beta}) d\boldsymbol{\beta}}{(1-\beta_s)^5}\hat{\mathbf{W}}(\boldsymbol{\beta})\delta(\omega-\omega_d). \tag{44}$$

To evaluate the feasibility of frequency resolved TS polarimetry, an expression similar to (44) but with the sixth power weighting factor was suggested without derivation in Ref. 8.

The specific values of the spectral Stokes vector components, $\mathbf{S}^{(s)}(\omega) = \hat{\mathbf{M}}(\omega,\mu,\theta)\cdot\mathbf{S}^{(i)}(\chi,\psi)$, are functions of five variables and require numerical integration that is beyond the analytical scope of this paper. First numerical results and experimental applications are discussed by Giudicotti *et al.* in Ref. 22. They extensively benchmark their code by computing the frequency resolved Mueller matrix elements, integrating them over the spectrum and comparing with the frequency integrated analytical results (41). Good consistency between frequency integrated numerical values and analytical predictions is reported in the paper. Thus, the exact relativistic expressions (41) can be used as a reliable tool for benchmarking and verification of numerical codes for frequency resolved TS polarization.

For conventional degree of polarization analysis (3), rigorous proof that $P \le 1$ is provided by Schwarz's inequality applied for the time averaged elements of the polarization matrix (1). A similar mathematical analysis and proof that $P(\omega) \le 1$ should be performed for the frequency-resolved degree of polarization (43). Note particularly that the spectral degree of polarization (43) does not depend on the width of the frequency interval $\Delta\omega$. Choosing a very narrow frequency interval leads to the limiting transition of a pure monochromatic electromagnetic wave which, by definition (see, for example, Ref. 16), is fully polarized with $P(\omega)\to 1$. Some restriction on the minimal value of $\Delta\omega$ or the conditions of applicability of $P(\omega)$ may be required to avoid an inconsistency. Analysis of this question should be a subject of future work.

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APPENDIX A: SCATTERING BY SINGLE ELECTRON

1. Stokes vector components

The Stokes parameters of the incident and scattered radiation are defined by time-averaged electric field components (1)

$$\begin{aligned}
S_0 &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(E_x(t)E_x^*(t) + E_t(t)E_t^*(t)\right)dt,\\
S_1 &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(E_x(t)E_x^*(t) - E_t(t)E_t^*(t)\right)dt,\\
S_2 &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(E_x(t)E_t^*(t) + E_t(t)E_x^*(t)\right)dt,\\
S_3 &= \lim_{T\to\infty}\frac{i}{2T}\int_{-T}^{T}\left(E_x(t)E_t^*(t) - E_t(t)E_x^*(t)\right)dt.
\end{aligned} \tag{A1}$$

They are determined by the projections of $\mathbf{E}(t)$ onto the $(\mathbf{e}_x,\mathbf{t}_i)$ and $(\mathbf{e}_x,\mathbf{t}_s)$ axes, respectively (see Equation (4)). Consider, for example, the Stokes vector $\mathbf{S}^{(i)}$ of the incident monochromatic wave (5)

$$\mathbf{E}_i = E_{ix}\mathbf{e}_x + E_{it}\mathbf{t}_i = E_0\left(\mathbf{e}'_x\cos\chi + i\mathbf{e}'_y\sin\chi\right), \tag{A2}$$

where the ellipticity angle $\chi = \pm\arctan(b_2/b_1)$. The two mutually perpendicular unit vectors $\mathbf{e}'_x$ and $\mathbf{e}'_y$ are orthogonal to the incident wave propagation direction $\mathbf{i}$. Their position with respect to the scattering plane is determined by the azimuth (orientation angle) ψ of the polarization ellipse ($\cos\psi = \mathbf{e}_x\cdot\mathbf{e}'_x$). Projecting (A2) onto the $\mathbf{e}_x$ and $\mathbf{t}_i$ axes gives the E_{ix} and E_{it} components

$$\begin{aligned}
E_{ix} &= \cos\chi\cos\psi - i\sin\chi\sin\psi,\\
E_{it} &= \cos\chi\sin\psi + i\sin\chi\cos\psi.
\end{aligned} \tag{A3}$$

Substituting (A3) in (A1) yields the Stokes vector of the fully polarized incident wave (2).

The particular case of a linearly polarized incident wave with $\chi = 0$ is considered in solution (19). The corresponding electric field amplitude $\mathbf{E}_i \parallel \mathbf{e}'_x$. The scattered wave propagation direction $\mathbf{s}$ is characterized in solution (19) by the polar and azimuth angles Θ and Φ. They are defined with respect to the spherical system of coordinates determined by the unit vectors $\hat{\mathbf{x}} = \mathbf{i}$, $\hat{\mathbf{y}} = \mathbf{e}'_x\times\mathbf{i}$, $\hat{\mathbf{z}} = \mathbf{e}'_x$. The relationships between the variables θ and ψ and the angles Θ and Φ are as follows:

$$\begin{aligned}
\cos\theta &= \mathbf{s}\cdot\mathbf{i} = \mathbf{s}\cdot\hat{\mathbf{x}} = \sin\Theta\cos\Phi,\\
\cos\psi &= \mathbf{e}'_x\cdot\mathbf{e}_x = \hat{\mathbf{z}}\cdot[\mathbf{i}\times\mathbf{s}]/\sin\theta = (\mathbf{s}\cdot\hat{\mathbf{y}})/\sin\theta\\
&= \sin\Theta\sin\Phi/\sin\theta.
\end{aligned} \tag{A4}$$

2. Spectral characteristics of the truncated fields

An infinitely long wave packet of incident monochromatic radiation (5) is characterized by a non-zero electric field from $-\infty \le t \le \infty$. The quadratic combinations of the scattered electric field (A1) are nonvanishing functions of time, while the Fourier expansion is formally valid only for functions decaying sufficiently fast at infinity (square integrable). In order to apply the Fourier transform to stationary scattered radiation, the method of truncated functions[15] is used

$$\mathbf{E}_s^{(T)}(t) = \begin{cases} \mathbf{E}_s(t) & |t| \le T \\ 0 & |t| \ge T \end{cases}, \tag{A5}$$

where t is the time at the remote detector (*time-at-observer*). We perform all intermediate calculations at finite T with the limiting transition in the final expressions.

The truncated signals are square integrable with the Fourier transform

$$\mathbf{E}_s^{(T)}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T}^{T} \mathbf{E}_s(t)\exp(i\omega t)dt. \tag{A6}$$

Consider a monochromatic dependence $\mathbf{E}_s(t) = \mathbf{E}_0\exp(-i\omega_d t)$ with Doppler shifted frequency $\omega_d = \omega_i(1-\beta_i)/(1-\beta_s))$ caused by Thomson scattering on a single electron with velocity $\mathbf{v}$. The Fourier image of the truncated field

$$\mathbf{E}_s^{(T)}(\omega) = \mathbf{E}_0\sqrt{\frac{2}{\pi}}\frac{\sin[(\omega-\omega_d)T]}{\omega-\omega_d} \tag{A7}$$

depends on ω and the truncation variable T. This is a smooth analytical function of ω at finite T and a singular δ-function at $T \to \infty$

$$\lim_{T\to\infty}\mathbf{E}_s^{(T)} = \mathbf{E}_s(\omega) = \mathbf{E}_0\sqrt{2\pi}\delta(\omega-\omega_d). \tag{A8}$$

For the general case of arbitrary time dependence, the truncated field is expressed by the inverse Fourier integral

$$\mathbf{E}_s^{(T)}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}_s^{(T)}(\omega)\exp(-i\omega t)d\omega. \tag{A9}$$

The time averaged quadratic combinations (A1) are expressed by double integrals over the frequencies ω_1 and ω_2. Consider, for example, the $S_0^{(s)}$ component

$$\begin{aligned} S_0^{(s)} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(E_{sx}(t)E_{sx}^\star(t) + E_{st}(t)E_{st}^\star(t)\right)dt \\ &= \lim_{T\to\infty}\frac{1}{2\pi T}\times\int_{-\infty}^{+\infty}d\omega_1\int_{-\infty}^{+\infty}d\omega_2\Big(E_{sx}^{(T)}(\omega_1)E_{sx}^{(T)\star}(\omega_2) \\ &\quad + E_{st}^{(T)}(\omega_1)E_{st}^{(T)\star}(\omega_2)\Big)\frac{\sin[(\omega_2-\omega_1)T]}{\omega_2-\omega_1}. \end{aligned} \tag{A10}$$

At sufficiently large T, the integral kernel is approximated by the δ-function

$$\frac{\sin[(\omega_2-\omega_1)T]}{\omega_2-\omega_1}\underset{T\to\infty}{\to}\pi\delta(\omega_2-\omega_1). \tag{A11}$$

Performing integration over ω_1 or ω_2, the time averaged quadratic combinations take the form of an integral over the spectrum

$$\begin{aligned} S_0^{(s)} &= \int_{-\infty}^{+\infty}d\omega S_0(\omega), \quad S_0(\omega) = \lim_{T\to\infty}\frac{S_0^{(s)}(\omega,T)}{2T}, \\ S_0^{(s)}(\omega,T) &= E_{sx}^{(T)}(\omega)E_{sx}^{(T)\star}(\omega) + E_{st}^{(T)}(\omega)E_{st}^{(T)\star}(\omega). \end{aligned} \tag{A12}$$

These transformations are equivalent to Parseval's theorem for spectral intensities. Using (A1), expression (A12) is generalized to the three other components of the Stokes vector in the form (11).

Specifying the dependence on T in the quadratic combinations (A12) allows us to perform the limiting transition $T \to \infty$ without uncertainties caused by the treatment of the square of a δ-function. Then, the expression for $S_0^{(s)}(\omega)$ takes a form

$$S_0^{(s)}(\omega) = \lim_{T\to\infty}\frac{|\mathbf{E}_0|^2}{\pi T}\frac{\sin^2[(\omega-\omega_d)T]}{(\omega-\omega_d)^2}. \tag{A13}$$

This function tends to zero at $\omega \ne \omega_d$ and to infinity at $\omega = \omega_d$, showing properties of the delta-function of ω. Exact integration over ω yields

$$\int_{-\infty}^{+\infty}S_0(\omega)d\omega = |\mathbf{E}_0|^2, \tag{A14}$$

indicating that

$$S_0^{(s)}(\omega) = |\mathbf{E}_0|^2\delta(\omega-\omega_d). \tag{A15}$$

3. Amplitude of the Thomson scattered field

Based on the truncation method, we calculate now the amplitude for the Thomson scattered field. The starting point is the Lienard-Wiechert expression for the scattered electric field $\mathbf{E}_s$ emitted by an electron moving along the unperturbed trajectory

$$\mathbf{R}(t') = \mathbf{R}_0 + \mathbf{v}t' \tag{A16}$$

and oscillating in the field (5) of the incident monochromatic wave $\boldsymbol{\mathcal{E}}_i(\mathbf{r}',t')$

$$\begin{aligned} \mathbf{E}_s(\mathbf{r},t) &= \frac{r_0}{r}\frac{\sqrt{1-\beta^2}}{(1-\beta_s)^2}\int d\mathbf{r}'\int dt'\delta[t'-t+(r-\mathbf{s}\cdot\mathbf{r}')/c] \\ &\quad\times\delta(\mathbf{r}'-\mathbf{R}(t'))\hat{\boldsymbol{\Pi}}\cdot\boldsymbol{\mathcal{E}}_i(\mathbf{r}',t'). \end{aligned} \tag{A17}$$

Note that we consider here an elementary process of scattering within the scope of the infinite scattering volume model treated in Secs. II A–II C of Ref. 12. The tensor $\hat{\boldsymbol{\Pi}}$ describes the transformation of polarization in the process of scattering on a single electron

$$\begin{aligned} \hat{\boldsymbol{\Pi}}\cdot\boldsymbol{\mathcal{E}}_i &= -(1-\beta_s)(1-\beta_i)\boldsymbol{\mathcal{E}}_i + [\beta_E(\cos\theta-\beta_s) \\ &\quad + (1-\beta_i)(\mathbf{s}\cdot\boldsymbol{\mathcal{E}}_i)]\mathbf{s} + [\beta_E(1-\cos\theta) \\ &\quad - (1-\beta_i)(\mathbf{s}\cdot\boldsymbol{\mathcal{E}}_i)]\boldsymbol{\beta} - \beta_E(1-\beta_s)\mathbf{i}, \end{aligned} \tag{A18}$$

where $\beta_E = \boldsymbol{\beta} \cdot \boldsymbol{\mathcal{E}}_i$. Equations (A17) and (A18) are also identical to the set of initial equations in Ref. 13.

The electric field $\mathbf{E}_s(\mathbf{r}, t)$ represents the scattered field at the remote position $\mathbf{r}$ on the detector at time t. We truncate the field $\mathbf{E}_s(\mathbf{r}, t)$ within a time interval $|t| \le T$ according to (A5). The Fourier image of the truncated signal is obtained by integrating over t from $-T$ to T

$$\mathbf{E}_s^{(T)}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{r_0}{r} \frac{\sqrt{1-\beta^2}}{(1-\beta_s)^2} \int d\mathbf{r}' \int dt' \delta[\mathbf{r}' - \mathbf{R}(t')] \times \hat{\boldsymbol{\Pi}} \cdot \boldsymbol{\mathcal{E}}_i(\mathbf{r}', t') \int_{-T}^{T} \delta\left[t' - t + (r - \mathbf{s} \cdot \mathbf{r}')/c\right] \times \exp(i\omega t)]dt, \tag{A19}$$

where unperturbed electron trajectories (A16) are used for integration. When t varies from $-T$ to T, the δ-function contributes to the integral if the retarded time t' is in the range $-T/(1-\beta_s) - t_r < t' < T/(1-\beta_s) - t_r$, where $t_r = (r - \mathbf{s} \cdot \mathbf{R}_0)/c(1-\beta_s)$ is the time shift due to the retardation and $\beta_s = \mathbf{s} \cdot \mathbf{v}/c$. In the exponential factor, t is expressed in terms of t' by equating to zero the argument of the δ-function. This determines the resulting integral over t'

$$\mathbf{E}_s^{(T)}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{r_0}{r} \frac{\sqrt{1-\beta^2}}{(1-\beta_s)^2} \int d\mathbf{r}' \int_{-T/(1-\beta_s)-t_r}^{T/(1-\beta_s)-t_r} dt' \times \delta[\mathbf{r}' - \mathbf{R}_0 - \mathbf{v}t'] \hat{\boldsymbol{\Pi}} \cdot \boldsymbol{\mathcal{E}}_i(\mathbf{r}', t') \times \exp\left[i\omega(1-\beta_s)t' + i\omega(r - \mathbf{s} \cdot \mathbf{R}_0)/c\right]. \tag{A20}$$

Using the explicit expression (5) for $\boldsymbol{\mathcal{E}}_i(\mathbf{r}', t')$ allows us to rewrite the integral over t' as follows:

$$\mathbf{E}_s^{(T)}(\omega) = \exp(i\omega_d r/c - i(\mathbf{k}_s - \mathbf{k}_i) \cdot \mathbf{R}_0) \frac{\hat{\boldsymbol{\Pi}} \cdot \mathbf{E}_i}{\sqrt{2\pi}} \frac{r_0}{r} \frac{\sqrt{1-\beta^2}}{(1-\beta_s)^2} \times \int_{-T/(1-\beta_s)}^{T/(1-\beta_s)} dt' \exp\left[i(\omega - \omega_i)t' - i(\omega \mathbf{s} - \omega_i \mathbf{i}) \cdot \boldsymbol{\beta} t'\right], \tag{A21}$$

where the new shifted time variable, $t' \to t' + t_r$, is used for integration while $\omega_d = \omega_i(1-\beta_i)/(1-\beta_s)$ is the Doppler shifted frequency and $\mathbf{k}_s = \omega_d \mathbf{s}/c$ is the wave vector of the scattered radiation. Performing integration over t' yields the final result in the form (8)

$$\mathbf{E}_s^{(T)}(\omega) = f^{(T)}(\omega, \boldsymbol{\beta}) \hat{\boldsymbol{\Pi}} \cdot \mathbf{E}_i,$$

where the scalar function $f^{(T)}(\omega, \boldsymbol{\beta})$

$$f^{(T)}(\omega, \boldsymbol{\beta}) = \exp(i\omega_d r/c - i(\mathbf{k}_s - \mathbf{k}_i) \cdot \mathbf{R}_0) \frac{r_0}{r} \frac{\sqrt{1-\beta^2}}{(1-\beta_s)^3} \times \sqrt{\frac{2}{\pi}} \frac{\sin(\Omega T)}{\Omega}, \quad \Omega = \omega - \omega_d \tag{A22}$$

is introduced for compact notation to describe the spectral characteristics and the dependences on r and $\boldsymbol{\beta}$. In the limiting case $T \to \infty$, the dependence on ω is expressed by the δ-function

$$\mathbf{E}_s(\omega) = \sqrt{2\pi} \exp(i\omega_d r/c - i(\mathbf{k}_s - \mathbf{k}_i) \cdot \mathbf{R}_0) \frac{r_0}{r} \frac{\sqrt{1-\beta^2}}{(1-\beta_s)^3} \times \hat{\boldsymbol{\Pi}} \cdot \mathbf{E}_i \delta(\omega - \omega_d). \tag{A23}$$

In this form, the result (A23) is consistent with Eq. (7) in Ref. 12.

The change of polarization is described by the polarization operator $\hat{\boldsymbol{\Pi}} \cdot \mathbf{E}_i$ given in (A18). Taking two orthogonal projections, the connection between the components of the scattered and incident electric fields is expressed by a 2×2 matrix (9). The matrix is obtained by projecting the scattered field on the $\mathbf{e}_x$ and $\mathbf{t}_s$ directions while the amplitude of the incident field $\mathbf{E}_i$ is projected on the $\mathbf{e}_x$ and $\mathbf{t}_i$ directions defined in (4). Then, the $\mathbf{e}_x$ component of the scattered electric field takes a form

$$E_{sx}(\omega)/f^{(T)}(\omega, \boldsymbol{\beta}) = -(1-\beta_s)(1-\beta_i)E_{ix} + [\beta_E(1-\cos\theta) + \sin\theta(1-\beta_i)E_{it}]\beta_x, \tag{A24}$$

while the $\mathbf{t}_s$ component is as follows:

$$E_{st}(\omega)/f^{(T)}(\omega, \boldsymbol{\beta}) = -\cos\theta(1-\beta_s)(1-\beta_i)E_{it} + (\beta_E(1-\cos\theta) + \sin\theta(1-\beta_i)E_{it})\beta_{ts} - \sin\theta\beta_E(1-\beta_s). \tag{A25}$$

The velocity components β_E, β_{ti}, and β_{ts} appearing in these relations are expressed in terms of β_i and β_s

$$\beta_E = \beta_x E_{ix} + \beta_{ti} E_{it}, \quad \beta_{ti} = \boldsymbol{\beta} \cdot \mathbf{t}_i = \frac{\beta_i \cos\theta - \beta_s}{\sin\theta}, \quad \beta_{ts} = \boldsymbol{\beta} \cdot \mathbf{t}_s = \frac{\beta_i - \beta_s \cos\theta}{\sin\theta}. \tag{A26}$$

Combining Equations (A24)–(A26) yields the matrix in Equation (9).

APPENDIX B: INFINITE (ITT) AND FINITE (FTT) TRANSIT TIME CASES

The scattering volume is defined by the intersection of the region occupied by the laser beam and the region of observation determined by the collection optics. The distortion of the signal caused by the finite size L of the scattering volume has a twofold effect. First, it broadens the spectrum (21) to the finite width $\delta\omega/\omega \simeq v_{Te}/(\omega L)$. The transit time broadening is much less than the expected thermal broadening in any high-temperature plasmas[1,12,14] and, therefore, ignored in our calculations. The second effect is less obvious and impacts the amplitude of the spectrum rather than its shape. It results in an additional factor $(1-\beta_s)$ in the numerator of the FTT intensity spectrum (24) compared to the ITT power spectrum (23). A quantitative physical picture of the FTT effect and origination of the additional factor were explained in Sec. III of Ref. 12 based on a single-bounded particle model for mean spectral intensity.

Revisiting this problem we developed a more intuitive picture of the nature of the FTT effect. It is based on consideration of a large number of electrons of a given velocity $\boldsymbol{\beta}$

crossing the scattering volume, visualization of the positions of the electromagnetic pulses emitted by the individual electrons, and counting the number of pulses passing through the remote detector at given time t on the detector. This geometrical picture shows that due to the combined effect of the retardation and electron motion, the number of pulses instantaneously passing through the remote detector is not equal to the stationary number of electrons residing in the scattering volume V, but to the number of electrons inside the modified volume $(1-\beta_s)V$. This justifies the use of the fifth power weighting factor for averaging not only the spectral intensity but also all components of the Stokes vector.

The appearance of the factor $(1-\beta_s)$ can also be interpreted in a different way: if the radiating particles are in a bounded volume the radiation intensity at the remote detector is determined not only by the single particle *time-at-observer* power $P^{(single)}$ but also by the *time-at-particle* power $P^{(particle)} = (1-\beta_s)P^{(single)}$ multiplied by the number of emitters inside the scattering volume V. For synchrotron radiation, the conclusion in this form was, for the first time, made in Ref. 20. A similar, but more succinct analysis of synchrotron radiation was presented, approximately at the same time, in Ref. 21. The analogy with synchrotron radiation was used in Ref. 2 to explain the need for the additional factor $(1-\beta_s)$ to account for the combined effect of many electrons.

1. Mueller matrix for averaging with the ITT weighting factor

The results presented in Sec. IV are obtained by averaging with the FTT weighting factor $\propto (1-\beta^2)/(1-\beta_s)^5$ that follows from Equation (25). To illustrate the importance of the specific form of the weighting factor, we apply the technique described in Sec. IV to the case of the weighting factor $\propto (1-\beta^2)/(1-\beta_s)^6$ that was used in Ref. 8 for calculations to the lowest linear order in $T_e/m_e c^2 \ll 1$. The analytical results for the sixth power weighting factor are represented by the Mueller matrix $\hat{\mathbf{h}}$ defined in Equation (42)

$$
\begin{aligned}
h_{00} &= 1 + u^2 + 2G(\mu)\left(5 - 6u + u^2 + \frac{84(1-u)^2}{\mu^2}\right) + \frac{4(17-9u)(1-u)}{\mu^2} + \frac{672(1-u)^2}{\mu^4},\\
h_{01} &= h_{10} = (1-u^2)\left(1 + 4G(\mu) + \frac{16}{\mu^2}\right),\\
h_{11} &= 1 + u^2 + 6\left(1 + \frac{24}{\mu^2}\right)(1-u)^2 G(\mu) + \frac{48(1-u)^2}{\mu^2} + \frac{576(1-u)^2}{\mu^4},\\
h_{22} &= -2u - 6\left(1 + \frac{24}{\mu^2}\right)(u-1)^2 G(\mu) - \frac{576(u-1)^2}{\mu^4} - \frac{48(u-1)^2}{\mu^2},\\
h_{33} &= 2u - 2G(\mu)\left(1 - 6u + 5u^2 + \frac{60(1-u)^2}{\mu^2}\right) + \frac{4(7-15u)(1-u)}{\mu^2} + \frac{480(1-u)^2}{\mu^4}.
\end{aligned}
\tag{B1}
$$

Comparing the two Mueller matrices $\hat{\mathbf{m}}$ and $\hat{\mathbf{h}}$ shows that they are very sensitive to the specific form of the weighting factor. For example, the off-diagonal elements $m_{01} = m_{10} = \sin^2\theta$ are always the same as in cold plasma while the equivalent elements of the matrix $\hat{\mathbf{h}}$ are substantial functions of electron temperature. While the specific form of the weighting factor is not important for the first integration over ϕ, it becomes critical at the second integration over α in Equation (33). Indeed, integrating the element $m_{01}^{(\phi)}$ given in Equation (32) over x with the fifth power weighting factor leads to cancellation of the $1-\beta^2$ terms in the numerator and denominator of the resulting expression. This corresponds to temperature independent off-diagonal elements $m_{01} = m_{10}$.

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