

# Rigorous treatment of charge exchange, ionization, and collisional processes in neutral-beam-injected mirrors

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(Received 12 April 1994; accepted 19 September 1994)

The ion distribution function is analytically studied in mirror machines with perpendicular injection. A uniform model is considered by assuming a square-well configuration for the magnetic field. The existence of the mirrors and the consequent electrostatic potential are represented by related boundary conditions on the ion distribution function. The Vlasov–Boltzmann equation is written with the explicit expressions for the charge exchange, electron impact ionization, ion–electron collision processes and solved for the steady state. The justification for neglecting the ion–ion collisions is provided. © 1995 American Institute of Physics.

## I. INTRODUCTION

The concept of a steady-state density in neutral-beam-injected mirror machines has been studied repeatedly<sup>1–6</sup> during the past 30 years. The kinetic studies were based on the Fokker–Planck equations involving nonlinear terms, and the problem had to be treated numerically in most of these works.<sup>1–4</sup> Furthermore, due to the complicated dependence of the atomic processes on the particle energies, or due to the inadequacy of data, available earlier on the associated cross sections, these terms were either treated formally or disregarded totally in the previous works.

In this paper, the relevant collisional and atomic processes are discussed in line with the experimental data, and treated in their explicit, actual forms. A uniform model is considered by imposing a square-well type of configuration for the magnetic field. The beam injection is assumed to be in the perpendicular direction. The ion velocity and electron temperatures are assumed to be in the range, which allows the ion drag on the electrons to emerge as the dominant collisional process. More explicitly, this corresponds to the case, where the electron temperature is low enough to let the drag time needed by the ion to travel from the source point localized at high energy, to the loss cone boundary localized at low energy to be shorter than the ion–ion scattering time. Consequently, the ion–ion collisions result in a slight broadening of the angular distribution, close to that inherent to the actual neutral beam, whose angular distribution is approximated to be a delta function in this work. The speed or energy dependence of the ion distribution function can therefore be analytically obtained by neglecting the diffusive terms due to ion–ion collisions, and thereby avoiding the nonlinear terms in the formal Fokker–Planck treatment. This distribution function can then be used to estimate the angular spread due to ion–ion collisions, and to verify that it is indeed a minor correction for the bulk of the function.

The losses through the mirrors are not explicitly dealt with, but instead introduced as a loss cone boundary condition on the ion distribution function. That is, the velocity dependence is regarded as a step-like function, which is zero for velocities inside the loss cone boundary, and finite

otherwise.<sup>5,6</sup> This step-like behavior is a consequence of neglecting the diffusive terms, which leads to the steepening of the ion distribution function around the loss cone boundary.

Following the prescribed procedure, the Vlasov–Boltzmann equation is solved analytically for the steady-state distribution function. A self-consistent expression for the steady-state ion density is also derived. Finally, the optimization of the plasma and beam parameters, to achieve maximum steady-state density is discussed.

## II. THE RELEVANT COLLISIONAL AND ATOMIC PROCESSES

The complete form of the Vlasov–Boltzmann equation for the ion distribution function can be formally written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{M} \cdot \frac{\partial f}{\partial \mathbf{v}} = -\nabla_{\mathbf{v}} \cdot \mathbf{J} + Q(\mathbf{v}), \quad (1)$$

where  $f$  and  $M$  denote the ion distribution function and the ion mass, respectively,  $\mathbf{J}$  is the ion flux in velocity space due to collisions, and  $Q(\mathbf{v})$  represents the source and loss terms for ions with velocity  $\mathbf{v}$ . In this section, the collisional and the atomic processes contributing to the right-hand side will be considered in detail.

According to Landau,<sup>7</sup> the  $i$ th component of the ion flux can be written as

$$J_i = 2\pi e^4 \Lambda M^{-1} \sum_{\beta, k} \left( \frac{f}{m_{\beta}} \int U_{ik} \frac{\partial f_{\beta}(\mathbf{v}')}{\partial v'_k} d^3 v' - M^{-1} \frac{\partial f}{\partial v_k} \int U_{ik} f_{\beta}(\mathbf{v}') d^3 v' \right), \quad (2)$$

where  $\Lambda$  is the Coulomb logarithm,  $\beta$  represents electrons and ions,  $k$  represents the components and

$$U_{ik} = \frac{|\mathbf{v} - \mathbf{v}'| \delta_{ik} - (v_i - v'_i)(v_k - v'_k)}{|\mathbf{v} - \mathbf{v}'|^3}. \quad (3)$$

To evaluate the contribution of electrons ( $\beta=e$ ) to the ion flux, the electron distribution function is assumed to be Maxwellian,

$$f_e = n \left( \frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left( -\frac{m_e v_e^2}{2T_e} \right), \quad (4)$$

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where  $n$  is the plasma density,  $m_e$ ,  $T_e$ , and  $v_e$  are the electron mass, temperature and velocity, respectively. After straightforward manipulations, the electron contribution to the  $i$ th component of the ion flux is obtained as

$$J_i^{(e)} = -2\pi e^4 \Lambda M^{-1} \times \sum_k \left( v_k \frac{f}{T_e} + M^{-1} \frac{\partial f}{\partial v_k} \right) \int U_{ik} f_e d^3 v_e. \quad (5)$$

Since the ion velocity is usually much less than the electron velocity, it can be shown that the term

$$\int U_{ik} f_e d^3 v_e \sim \int \left( \frac{\delta_{ik}}{v_e} - \frac{v_{ei} v_{ek}}{v_e^3} \right) f_e d^3 v_e,$$

and therefore

$$J_i^{(e)} \approx -\frac{4\pi e^4 \Lambda n}{3M} \left( \frac{2m_e}{\pi T_e} \right)^{1/2} \left( v_i \frac{f}{T_e} + M^{-1} \frac{\partial f}{\partial v_i} \right).$$

Concerning the component along  $\mathbf{v}$ , the second term can be neglected for  $v \gg (T_e/M)^{1/2}$ . The  $\theta$  component of  $\mathbf{J}^{(e)}$ , arising from the second term is much less than the  $\theta$  component of the flux due to ion-ion collisions (for ion velocities much less than electron thermal velocity). Since even the ion-ion collisions were explained to be negligible in the Introduction, the electron contribution to the ion flux can be written as

$$\mathbf{J}^{(e)} = -C \nabla f(\mathbf{v}), \quad (6)$$

where

$$C = \frac{4\pi e^4 \Lambda n}{3T_e M} \left( \frac{2m_e}{\pi T_e} \right)^{1/2}. \quad (7)$$

The contribution of the ion-ion collisions to the ion flux will be disregarded, due to the choice of the parameter range discussed in the Introduction. This contribution will be estimated later in the work, to verify that it is indeed a minor correction, concerning the bulk of the ion distribution function. Thus Eq. (6) is assumed to give the total ion flux at this stage. We shall now proceed with the relevant atomic processes, contributing to  $Q(\mathbf{v})$  on the right-hand side of Eq. (1).

The first process to be considered is the charge exchange between the neutral beam and the plasma ions. At a particular magnetic field surface, the neutral beam having a density  $n_b$  (less than the injection density, due to ionization) yields ions with the beam velocity  $\mathbf{v}_b$ , upon charge exchange with the plasma ions. The rate, at which the ion distribution function increases at  $\mathbf{v} = \mathbf{v}_b$  due to charge exchange can be written as

$$Q_{\text{ex}}^+(\mathbf{v}) = n_b \delta(\mathbf{v} - \mathbf{v}_b) \int \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) f(\mathbf{v}) |\mathbf{v} - \mathbf{v}_b| d^3 v, \quad (8)$$

where  $\sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|)$  is the charge-exchange cross section, which is a function of the relative velocity. On the other hand, the plasma ions neutralized by the beam via charge exchange, escape from the system, yielding a rate of loss for the whole ion distribution  $f(\mathbf{v})$ , given by

$$Q_{\text{ex}}^-(\mathbf{v}) = -n_b \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) f(\mathbf{v}) |\mathbf{v} - \mathbf{v}_b|. \quad (9)$$

The next process to be considered is the ionization of the neutral beam, due to the electron and ion impact. Since electrons are clearly much more dominant, the latter will be ignored. This process also yields ions with the beam velocity, and the corresponding rate at which the ion distribution function increases can be written as

$$Q_i(\mathbf{v}) = n_b \delta(\mathbf{v} - \mathbf{v}_b) \int f_e(\mathbf{v}_e) \sigma_i(|\mathbf{v}_e - \mathbf{v}_b|) |\mathbf{v}_e - \mathbf{v}_b| d^3 v_e, \quad (10)$$

where  $\sigma_i(|\mathbf{v}_e - \mathbf{v}_b|)$  is the cross section for the electron impact ionization. Since  $v_b \ll v_e$ ,

$$Q_i(\mathbf{v}) \approx n_b \delta(\mathbf{v} - \mathbf{v}_b) \int f_e(\mathbf{v}_e) \sigma_i(v_e) v_e d^3 v_e = n_b n \delta(\mathbf{v} - \mathbf{v}_b) \langle \sigma_i v_e \rangle. \quad (11)$$

It must be noted that  $\langle \sigma_i v_e \rangle$  is the ionization rate coefficient due to electron impact, which depends on the electron temperature  $T_e$  only. Although the electron temperature in mirror machines depends on the ion density and energy, as well as other parameters, it will be regarded as a given, constant quantity in this work.

### III. THE STEADY-STATE SOLUTION

The complete steady-state form of Eq. (1) for a uniform model can be written as,

$$\frac{e}{Mc} (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} = -\nabla_{\mathbf{v}} \cdot \mathbf{J}^{(e)} + Q(\mathbf{v}). \quad (12)$$

The term  $\mathbf{J}^{(e)}$  represents the ion flux due to electron-ion drag. The explicit form of  $Q(\mathbf{v})$ , in principle consists of the sum of the charge exchange and ionization rates discussed in the previous section, and the loss rate through the mirrors. This loss rate will not be considered explicitly in the term  $Q(\mathbf{v})$ , but will be taken into account as a loss cone boundary, on the surface of which, the ion distribution drops to zero. In the standard approach,<sup>6</sup> the ion distribution function is assumed to drop to zero gradually. However, since the diffusive terms are ignored in this work, the distribution function steepens, forming a more or less step function discontinuity at the boundary. This profile does not violate the continuity of the ion current, since it is determined by the gradient of the distribution function at the very thin transition layer (diffusive terms), rather than its magnitude.

Returning to Eq. (12), letting the magnetic field be in the  $z$  direction and substituting Eqs. (6), (8), (9), and (11), one can write

$$-\omega_c \frac{\partial f}{\partial \varphi} = C \nabla_{\mathbf{v}} \cdot (\mathbf{v} f) + n_b \delta(\mathbf{v} - \mathbf{v}_b) \left[ n \langle \sigma_i v_e \rangle + \int \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) |\mathbf{v} - \mathbf{v}_b| f d^3 v - n_b \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) |\mathbf{v} - \mathbf{v}_b| f \right], \quad (13)$$

where  $\omega_c = eB/Mc$  and  $\varphi$  is the azimuthal angle. Substituting typical values for mirror machines, it can be seen that  $\omega_c$  is much larger than the frequency of collisions and charge exchange, implying that  $\partial f / \partial \varphi$  must be very small. Due to

the periodicity of variations with respect to  $\varphi$ , this can be possible only if  $f$  consists of a large  $\varphi$  independent part and a small  $\varphi$  dependent part. Therefore,

$$f(\mathbf{v}) = f_0(\mathbf{v}) + f_1(\mathbf{v}),$$

where  $f_0(\mathbf{v}) \gg f_1(\mathbf{v})$ . Using this expansion, the zeroth-order form of Eq. (13) confirms the fact that  $f_0(\mathbf{v})$  is independent of  $\varphi$ , and the first-order form yields

$$\begin{aligned} -\omega_c \frac{\partial f_1}{\partial \varphi} = & C \nabla_{\mathbf{v}} \cdot (\mathbf{v} f_0) + n_b \delta(\mathbf{v} - \mathbf{v}_b) \left[ n \langle \sigma_i v_e \rangle \right. \\ & \left. + \int \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) |\mathbf{v} - \mathbf{v}_b| f_0 d^3 v \right] \\ & - n_b \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) |\mathbf{v} - \mathbf{v}_b| f_0. \end{aligned} \quad (14)$$

We shall now adopt the spherical coordinates in velocity space, with  $\varphi$  remaining as the azimuthal angle. Setting the beam velocity arbitrarily in  $x$  direction for perpendicular injection, one can write

$$\delta(\mathbf{v} - \mathbf{v}_b) = v_b^{-2} \delta(v - v_b) \delta(\varphi) \delta(\theta - \pi/2).$$

Using this expression and taking the average value of Eq. (14) with respect to angle  $\varphi$ , the left-hand side vanishes and one obtains

$$\frac{\partial}{\partial v} (v^3 f_0) - g(v, \theta) (v^3 f_0) = A v^2 \delta(v - v_b) \delta(\theta - \pi/2), \quad (15)$$

where

$$g(v, \theta) = \frac{n_b}{2\pi C v} \int_0^{2\pi} \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) |\mathbf{v} - \mathbf{v}_b| d\varphi \quad (16)$$

and

$$\begin{aligned} A = & -\frac{n_b}{2\pi C v_b^2} \left( n \langle \sigma_i v_e \rangle \right. \\ & \left. + \int \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|) f_0(\mathbf{v}) |\mathbf{v} - \mathbf{v}_b| d^3 v \right). \end{aligned} \quad (17)$$

Equation (15) can be solved for the regions  $v < v_b$  and  $v > v_b$  separately. Both solutions are of the form

$$f_0(v, \theta) = \frac{K(\theta)}{v^3} \alpha(v, \theta), \quad (18)$$

where  $K(\theta)$  is an arbitrary function with two different values for  $v < v_b$  and  $v > v_b$  regions, and

$$\alpha(v, \theta) = \exp \int g(v, \theta) dv. \quad (19)$$

The steady-state density involves the integration of  $f_0(v, \theta)$  in spherical coordinates, that is, the integral of  $v^{-1} \alpha(v, \theta) dv$ . Since  $g(v, \theta)$  is always a positive function of  $v$ ,  $\alpha(v, \theta) > 1$  and this integral will diverge at the upper limit,  $v = +\infty$ . For a finite steady-state density, the arbitrary function  $K(\theta)$  must therefore be zero for  $v > v_b$ , hence

$$f_0(v, \theta) = \begin{cases} 0, & \text{for } v < v_0, \\ K(\theta) v^{-3} \alpha(v, \theta), & \text{for } v_0 < v < v_b, \\ 0, & \text{for } v > v_b, \end{cases} \quad (20)$$

where  $v_0$  represents the velocity at the loss cone boundary.

To evaluate the function  $K(\theta)$ , Eq. (15) is integrated over the velocity between the limits  $v = v_b \pm \epsilon$ , where  $\epsilon$  is arbitrarily small. Substituting Eq. (20), this procedure yields

$$K(\theta) = -\frac{A v_b^2 \delta(\theta - \pi/2)}{\alpha(v_b, \theta)}. \quad (21)$$

Equation (21) indicates that the ion distribution is a disk-like distribution localized precisely at  $\theta = \pi/2$ , implying that all velocities including  $v_0$  are purely in the perpendicular direction. This may appear to contradict with the existence of mirror losses. At this point, it is necessary to remember that, this form of the distribution function is the consequence of neglecting the ion-ion collisions, and it is hence an approximation. In reality, there is a small angular spread to be given by Eq. (51), which justifies using the hyperbolic form of the loss cone boundary, where  $v_0$  is basically determined by the ambipolar potential.

Equations (20) and (21) constitute the formal solution for the ion distribution function. To obtain the explicit form, we shall start with investigating the constant term  $A$ . Substituting Eqs. (20) and (21), Eq. (17) can be rewritten as

$$\begin{aligned} A = & -\frac{n_b n \langle \sigma_i v_e \rangle}{2\pi C v_b^2} \\ & + \frac{n_b A}{2\pi C} \int \frac{\delta(\theta - \pi/2) \alpha(v, \theta) |\mathbf{v} - \mathbf{v}_b| \sigma_{\text{ex}}(|\mathbf{v} - \mathbf{v}_b|)}{v \alpha(v_b, \theta)} \\ & \times \sin \theta d\theta d\varphi dv. \end{aligned} \quad (22)$$

After integrating over  $\theta$ , the integral in this equation takes the form

$$\begin{aligned} I = & \alpha^{-1}(v_b, \pi/2) \int_{v_0}^{v_b} dv v^{-1} \alpha(v, \pi/2) \\ & \times \int_0^{2\pi} v_{\text{rel}} \sigma_{\text{ex}}(v_{\text{rel}}) d\varphi, \end{aligned} \quad (23)$$

where

$$v_{\text{rel}} = |\mathbf{v} - \mathbf{v}_b|_{\theta=\pi/2} = (v^2 + v_b^2 - 2vv_b \cos \varphi)^{1/2}. \quad (24)$$

It can be seen from Eq. (16) that, the integration over  $\varphi$  in Eq. (23) is simply  $2\pi C v g(v, \pi/2) / n_b$ . Using this fact and Eq. (19), Eq. (23) can be written as

$$\begin{aligned} I = & \frac{2\pi C / n_b}{\alpha(v_b, \pi/2)} \int_{v_0}^{v_b} \left[ \exp \int g(v, \pi/2) dv \right] g(v, \pi/2) dv \\ = & \frac{2\pi C [\alpha(v_b, \pi/2) - \alpha(v_0, \pi/2)]}{n_b \alpha(v_b, \pi/2)}. \end{aligned} \quad (25)$$

Substituting this result into Eq. (22) yields

$$A = -\frac{n_b n \langle \sigma_i v_e \rangle \alpha(v_b, \pi/2)}{2\pi C v_b^2 \alpha(v_0, \pi/2)}. \quad (26)$$

Using Eqs. (20), (21), and (26), the ion distribution function for  $v_0 < v < v_b$  can now be written as

$$f_0(v, \theta) = \left( \frac{n_b n \langle \sigma_i v_e \rangle \alpha(v_b, \pi/2)}{2\pi C \alpha(v_0, \pi/2)} \right) \frac{\alpha(v, \theta) \delta(\theta - \pi/2)}{\alpha(v_b, \theta) v^3} \quad (27)$$

The next task is to derive the explicit form of the function  $\alpha(v, \theta)$  or simply  $\alpha(v, \pi/2)$ , due to the delta function in Eq. (27). An analytic expression for the charge exchange cross section  $\sigma_{ex}$  in the function  $\alpha(v, \pi/2)$ , can be derived from the experimental data<sup>8</sup> by curve fitting techniques as

$$\sigma_{ex} = 10^{-14} (1 - 0.5E^{0.06} + 2 \cdot 10^{-7} E) \text{ cm}^2, \quad (28)$$

where  $E$  is the collision energy in eV for deuterium ions and neutrals, that is,

$$E = 1.04 \cdot 10^{-12} |\mathbf{v} - \mathbf{v}_b|^2. \quad (29)$$

Equation (28) agrees very well with the data, up to energies on the order of 400 keV, corresponding to particle energies of 100 keV, which is applicable to all present technologies. Substituting Eqs. (28) and (29) into Eq. (16) yields

$$g(v, \pi/2) = \frac{10^{-14} n_b}{2\pi C v} \int_0^{2\pi} v_{rel} (1 - 0.095 v_{rel}^{0.12} + 2.08 \cdot 10^{-19} v_{rel}^2) d\varphi, \quad (30)$$

where  $v_{rel}$  is the term defined by Eq. (24). The integral in Eq. (30) can be evaluated numerically for a set of values of  $(v/v_b)$ , and the following polynomial is found to represent  $g(v, \pi/2)$  with an accuracy better than 95%:

$$g(v, \pi/2) = (n_b/C) (a_0 \xi^{-1} + a_1 + a_2 \xi + a_3 \xi^2), \quad (31)$$

where  $\xi = v/v_b$ ,

$$\frac{(v_0/v_b)^{\lambda v_b a_0 (n_b/n)} \exp\{\lambda v_b (n_b/n) [a_1 (v_0/v_b) + (a_2/2)(v_0/v_b)^2 + (a_3/3)(v_0/v_b)^3]\}}{\lambda \langle \sigma_i v_e \rangle (n_b/n)}$$

$$= \int_{v_0/v_b}^1 \xi^{\lambda v_b a_0 (n_b/n) - 1} \exp\left\{ \lambda v_b \left( \frac{n_b}{n} \right) \left[ a_1 \xi + \left( \frac{a_2}{2} \right) \xi^2 + \left( \frac{a_3}{3} \right) \xi^3 \right] \right\} d\xi, \quad (34)$$

where  $\lambda = n/C$  is substituted to distinguish the  $n$  dependent terms. To explore the possibility of expanding the exponential functions in this equation, the coefficients  $a_i$ , will be investigated. For a wide range of beam energy from 10 to 100 keV, the ranges of these coefficients are found to be  $a_0 \sim 10^{-14} (0.13 - 0.03)$ ,  $a_1 \sim 10^{-15} (0.2 - 0.3)$ ,  $a_2 \sim 10^{-15} (0.03 - 0.12)$  and  $a_3 \sim 10^{-16} (0.1 - 0.9)$ . Since  $\xi \leq 1$  and  $(v_0/v_b) \ll 1$ , the exponents are always negative. It can then be seen that, the left-hand side of Eq. (34) starts at  $+\infty$  for  $(n_b/n) = 0$ , decays very rapidly and asymptotically goes to zero for large  $(n_b/n)$ . On the other hand, the right-hand side of Eq. (34) starts with a value of  $\ln(v_b/v_0)$  at  $(n_b/n) = 0$ , decays smoothly (with a finite slope, on the order of  $\lambda v_b a_0$  at small  $n_b/n$ ) and asymptotically goes to zero also, for large  $(n_b/n)$ . Furthermore, the slope of the left-hand side is more negative than the slope of the right-hand side, definitely for small and large values of  $(n_b/n)$ , and very

$$a_0 = 10^{-14} (1 - 0.095 v_b^{0.12} + 2.08 \cdot 10^{-19} v_b^2),$$

$$a_1 = 10^{-14} (0.048 - 7.6 \cdot 10^{-3} v_b^{0.12}),$$

$$a_2 = 10^{-14} (0.23 - 0.026 v_b^{0.12} + 4.08 \cdot 10^{-19} v_b^2),$$

$$a_3 = 1.1 \cdot 10^{-33} v_b^2.$$

Substituting Eq. (31) into Eq. (19) yields

$$\alpha(\xi, \pi/2) = \xi^{(n_b v_b a_0 / C)} \exp\left\{ \left( \frac{n_b v_b}{C} \right) \left[ a_1 \xi + \left( \frac{a_2}{2} \right) \xi^2 + \left( \frac{a_3}{3} \right) \xi^3 \right] \right\}, \quad (32)$$

and the ion distribution, given by Eq. (27), can now be written in its final, explicit form as

$$f_0(v) = \frac{n_b n \langle \sigma_i v_e \rangle}{2\pi C v^3} \left( \frac{v}{v_0} \right)^{(n_b v_b a_0 / C)} \exp\left\{ \left( \frac{n_b v_b}{C} \right) \times \left[ a_1 \left( \frac{v - v_0}{v_b} \right) + \frac{a_2}{2} \left( \frac{v^2 - v_0^2}{v_b^2} \right) + \frac{a_3}{3} \left( \frac{v^3 - v_0^3}{v_b^3} \right) \right] \right\}. \quad (33)$$

It is to be remembered that, this function exists only for  $\theta = \pi/2$  and  $v \leq v_b$ .

To complete the steady-state analysis, we shall now solve for the density  $n$  in a self-consistent way. Integrating the function  $f_0(v)$  over the three-dimensional velocity space, from  $v = v_0$  to  $v = v_b$  and equating to  $n$  yields

likely for the entire range of this parameter. This picture implies that, there is only one solution of Eq. (34) for  $(n_b/n)$ , and it is a very small quantity. With this intuition, the exponents in Eq. (34) are approximated as unity, and the following expression for the steady-state density is obtained:

$$n \approx \lambda v_b a_0 n_b \ln(v_b/v_0) / \ln\left( 1 + \frac{v_b a_0}{\langle \sigma_i v_e \rangle} \right). \quad (35)$$

Having thus completed the steady-state analysis, the optimum values of the controllable parameters in this equation will now be briefly discussed. The factor  $\lambda$  is on the order of  $10^8 T_e^{3/2}$  (eV). For the electron impact ionization rate, the expression given in Ref. 9 is adopted:

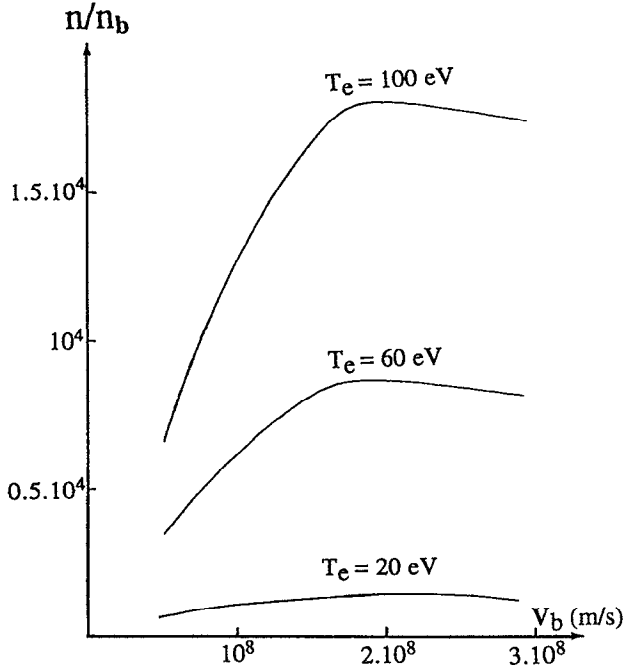


FIG. 1. Dependence of  $(n/n_b)$  on the beam velocity  $v_b$  and electron temperature  $T_e$ .

$$\langle \sigma_i v_e \rangle \approx 5.3 \cdot 10^{-8} \left( z^{1/2} E(z) - \frac{1.05 z^{3/2} E(z+0.56)}{z+0.56} \right) \text{ cm}^3 \text{ s}^{-1}, \quad (36)$$

where  $z = 13.6/T_e$  (eV) and  $E(z)$  is the exponential integral. According to this expression, the ionization rate is on the order of  $10^{-8} \text{ cm}^3 \text{ s}^{-1}$  at  $T_e = 20 \text{ eV}$  and gradually increases upto  $3 \cdot 10^{-8} \text{ cm}^3 \text{ s}^{-1}$  at  $T_e \sim 150 \text{ eV}$ , which covers adequately the typical range for the mirror machines. The lower bound for the velocity  $v_0$  has a rather complicated dependence on the electron temperature. It can be estimated from the electrostatic potential, to correspond roughly to an energy, on the order of  $T_e \ln(M/m_e)^{1/2}$ . A more rigorous treatment of this quantity is not necessary, since the term  $\ln(v_b/v_0)$  is found to vary between 2 and 3, for a broad range of beam energy from 10 to 100 keV, and electron temperatures from 20 to 100 eV. It can then safely be stated that the steady-state density increases with the electron temperature and the beam density. To explore the dependence on the beam velocity, the product  $v_b a_0$  should be considered together, since  $a_0$  is a function of  $v_b$ . Treating the term  $\ln(v_b/v_0)$  as a constant, it can easily be shown that the steady-state density increases with the term  $v_b a_0$ . Using Eq. (31),  $v_b a_0$  is found to increase with  $v_b$ , up to a value of  $1.35 \cdot 10^{-7}$  at  $v_b = 10^8 \text{ cm s}^{-1}$ , then decrease until  $v_b = 6.5 \cdot 10^8 \text{ cm s}^{-1}$ , and increase again for larger values of  $v_b$ . However, since the latter value of  $v_b$  already corresponds to a beam energy of 400 keV, the optimum value of the beam velocity should be around  $10^8 \text{ cm s}^{-1}$  for the present technology. Then, choosing the optimum values as  $T_e \sim 80 \text{ eV}$  and  $v_b = 10^8 \text{ cm s}^{-1}$ , the maxi-

imum value of the steady-state density is found to be on the order of  $10^4 n_b$ . The dependence of  $(n/n_b)$  on  $v_b$  is illustrated in Fig. 1, for a set of values of  $T_e$ .

As a final point in this work, we shall now attempt to provide justifications for the two assumptions made so far; equating the exponential functions in Eq. (34) to approximately unity, and neglecting the ion-ion collisions in Eq. (12). For the optimum values discussed above, it can be seen from Eqs. (31) and (35) that  $|a_1/a_0| \sim 0.14$  and  $\lambda v_b a_0 (n_b/n) < 1$ , respectively. The magnitudes of the exponents in Eq. (34) are roughly  $\lambda v_b |a_1| (n_b/n) (v_0/v_b)$  and  $\lambda v_b |a_1| (n_b/n) \xi$ , for the left- and right-hand sides, respectively. Since  $v_0 \ll v_b$  and  $\xi$  is considerably less than unity for a large range of the integral in Eq. (34), the maximum error resulting from the first assumption is less than 10%, which is insignificant.

As far as the ion-ion collisions are concerned, the derivations are rather detailed and only the main points will be stated. The contribution of ion-ion collisions to the right-hand side of Eq. (12) can be written as

$$-\nabla_v \cdot \mathbf{J}^{(i)} = -v^{-2} \frac{\partial}{\partial v} (v^2 J_v^{(i)}) - \frac{1}{v \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta J_\theta^{(i)}). \quad (37)$$

Due to the expectation that the ion distribution has a very narrow angular spread ( $\delta$  function in our former results), the second term on the right-hand side of Eq. (37) is dominant. Starting with Eq. (2), one finally obtains the following expression for this term:

$$J_\theta^{(i)} = C' \int d^3 v' \left[ g_\theta \left( \frac{v'^2 a_{\theta v'}^2}{u^3} - \frac{1}{u} \right) + g'_\theta \left( \frac{a_{\theta \theta'}}{u} + \frac{v v' a_{\theta v'} a_{v \theta'}}{u^3} \right) + G_{vv'} \frac{a_{\theta v'}}{u^3} (v - a_{vv'} v') v' \right], \quad (38)$$

where  $C' = 2\pi \Lambda e^4 / M^2$ ,  $g_\theta = (f'/v) \partial f / \partial \theta$ ,  $g'_\theta = (f'/v') \partial f' / \partial \theta'$ ,  $G_{vv'} = (v f / v') \partial f' / \partial v' - f' (\partial f / \partial v)$ ,  $a_{\theta v'} = \sin \theta' \cos \theta \times \cos(\varphi - \varphi') - \cos \theta' \sin \theta$ ,  $a_{v \theta'} = \sin \theta \sin \theta' \cos(\varphi - \varphi')$ ,  $a_{\theta \theta'} = \sin \theta \cos \theta' \cos(\varphi - \varphi') - \cos \theta \sin \theta'$ ,  $a_{\theta v'} = \sin \theta \sin \theta' + \cos \theta \cos \theta' \cos(\varphi - \varphi')$  and  $u = (v^2 + v'^2 - 2v v' a_{vv'})^{1/2}$ .

The steep derivative  $\partial f' / \partial \theta'$  in  $g'_\theta$  will be smoothed after the  $d\theta'$  integration, and  $G_{vv'}$  is a well-behaving function. Therefore, the main contribution is expected from the first term, that is,

$$J_\theta^{(i)} \approx \frac{C'}{v} \cdot \frac{\partial f}{\partial \theta} \int f' \left( \frac{v'^2 a_{\theta v'}^2}{u^3} - \frac{1}{u} \right) d^3 v'. \quad (39)$$

Again, since  $f$  and  $f'$  are expected to be highly peaked around  $\theta = \pi/2$  and  $\theta' = \pi/2$ , the contribution of the first term on the right-hand side of Eq. (39) will be very small, yielding

$$J_\theta^{(i)} \approx -\frac{C'}{v} \cdot \frac{\partial f}{\partial \theta} \int \frac{f'}{u} d^3 v'. \quad (40)$$

Accepting this expression as the total ion flux due to ion-ion collisions, substituting into Eq. (12), considering the

assumption that the ion distribution is highly peaked around  $\theta \sim \pi/2$ , and following the steps until Eq. (15), one obtains

$$\left[ \frac{C'}{Cv^3} \int \frac{f'}{u} d^3v' \right] \frac{\partial^2(v^3 f)}{\partial \theta^2} + \frac{\partial}{\partial v} (v^3 f) - g(v, \theta)(v^3 f) = Av^2 \delta(v - v_b) \delta(\theta - \pi/2). \quad (41)$$

Using  $g(v, \pi/2)$  instead of  $g(v, \theta)$  and letting

$$v^3 f = F \exp \int g(v, \pi/2) dv, \quad (42)$$

Eq. (41) takes the form,

$$H(v, \theta) \frac{\partial^2 F}{\partial \theta^2} + \frac{\partial F}{\partial v} = \frac{Av^2 \delta(v - v_b) \delta(\theta - \pi/2)}{\alpha(v, \pi/2)}, \quad (43)$$

where

$$H(v, \theta) = \frac{C'}{Cv^3} \int \frac{f'}{u} d^3v'. \quad (44)$$

Again, using  $H(v, \pi/2)$  instead of  $H(v, \theta)$  and letting

$$w(v) = \int_v^{v_b} H(v', \pi/2) dv', \quad (45)$$

Eq. (43) can be written as

$$\frac{\partial^2 F}{\partial \theta^2} \frac{\partial F}{\partial w} = \frac{Av^2 \delta(v - v_b) \delta(\theta - \pi/2)}{H(v, \pi/2) \alpha(v, \pi/2)}. \quad (46)$$

This is a diffusion type of equation with the solution of the form,

$$F = w^{-1/2} F_0 \exp[-(\theta - \theta_0)^2 / 4w], \quad (47)$$

where  $F_0$  and  $\theta_0$  are constants, which can be determined by integrating Eq. (43) over  $v$ , between the limits  $v = v_b \pm \epsilon$ , where  $\epsilon$  is arbitrarily small. This procedure yields

$$F(v_b + \epsilon) - F(v_b - \epsilon) = \frac{Av_b^2 \delta(\theta - \pi/2)}{\alpha(v_b, \pi/2)}. \quad (48)$$

Since  $F(v)$  is proportional to  $f_0$ , which is zero for  $v > v_b$ ,  $F(v_b + \epsilon) = 0$ , and

$$F(v_b) = - \frac{Av_b^2 \delta(\theta - \pi/2)}{\alpha(v_b, \pi/2)}. \quad (49)$$

Returning to Eq. (47) and noting that  $w(v_b) = 0$ , one can write

$$F(v_b) = 2\pi^{1/2} F_0 \cdot \delta(\theta - \theta_0). \quad (50)$$

Comparing Eqs. (49) and (50) yields

$$F_0 = - \frac{Av_b^2}{2\pi^{1/2} \alpha(v_b, \pi/2)} \quad \text{and} \quad \theta_0 = \pi/2.$$

Substituting this result into Eq. (47) and using Eq. (42), one finally obtains

$$f(v, \theta) = - \frac{Av_b^2 \alpha(v, \pi/2)}{v^3 \alpha(v_b, \pi/2)} [4\pi w(v)]^{-1/2} \times \exp[-(\theta - \pi/2)^2 / 4w(v)]. \quad (51)$$

It can easily be shown that, neglecting the ion-ion collisions corresponds to the limit  $w=0$ , and the previous result is recovered exactly. The effect of these collisions is then merely the broadening of the ion distribution function around  $\theta = \pi/2$ . This broadening is obviously negligible for the higher velocities, since  $w(v_b) = 0$ . We shall now consider its maximum value,  $w(v_0)$ . Using Eqs. (44) and (45), substituting the former expression of  $f(v)$  for simplicity, and integrating numerically yields

$$w(v_0) \sim \frac{10n_b T_e^{3/2} (\text{eV})}{n}. \quad (52)$$

For mirror machines, this quantity is considerably less than one, implying that the effect of ion-ion collisions on the ion distribution function can be neglected for a large range of velocities, except the relatively narrow range  $v \sim v_0$ , justifying our second assumption.

#### IV. DISCUSSION OF THE RESULTS

In this work, the Vlasov-Boltzmann equation is written with the explicit expressions of all the relevant collisional and atomic processes in neutral-beam-injected mirrors, and solved analytically for the steady-state ion distribution function. A self-consistent, analytic expression for the steady-state density is also obtained, by integrating the distribution function over the velocity space. The nonlinear ion-ion collision term is treated as a perturbation, and its contribution is later shown to be a slight broadening of the distribution function around the perpendicular direction, which gets noticeable only around  $v \sim v_0$ . The steady-state density is found to increase linearly with the beam density and logarithmically with the ratio of the beam velocity to the escape velocity. The dependence on the escape velocity is therefore relatively weak, but it is worth noting that the steady-state density becomes zero when these velocities are equal, and becomes infinite when the escape velocity is zero, as expected. Both the charge-exchange and electron impact ionization contribute positively to the steady-state density, where the dependence on the latter process is again relatively weak, due to the logarithmic behavior. The contribution of electron temperature is also positive, since it increases the ionization and decreases the electron drag on the ions. Finally, the steady-state density is found to increase with the beam velocity up to a certain value, but decrease for any further increase in the beam velocity, due to the fact that the charge-exchange cross section decreases rapidly with the relative velocity, after this quantity exceeds a certain value.

It can be seen that the results are in agreement with the physical expectations, but no comparison with the previous works can be made, since the atomic processes have not been considered with their explicit, exact forms before.

#### ACKNOWLEDGMENTS

The authors are indebted to the Turkish Scientific and Technical Research Council (TUBITAK) for initiating Professor V. Mirnov's visit to the Middle East Technical Univer-

sity and would also like to acknowledge that this work was supported, in part, by a Soros Foundation Grant awarded by the American Physical Society.

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